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VOL. XIX BATON ROUGE, LA., March, 1945 No. 6

Once More---The Culture Universal

Matrics and Quadric Surfaces

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A Mathematical Support for the Theory of the Conservation of Energy

A Tenth Lesson in the History of Mathematics

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Once More---The Culture Universal

In E. T. Bell's fascinating volume *Men of Mathematics* (Simon and Shuster, New York, 1937) are three pages of pronouncements on mathematics from thirty-five distinguished writers—most of them mathematicians. One of them, Query 64, quoted from Bishop Berkeley, asks whether "mathematicians have their mysteries, and, what is more, their contradictions?" Interestingly enough the "contradictions" hinted by the Query appears slightly borne out by the quotations (Plato's) "God ever geometrizes" and (Jacobi's) "God ever arithmetizes". Then, A. N. Whitehead's deliverance "When a mathematical or philosophical author writes with a 'misty profundity' it is safe to assume he is talking nonsense", may lead a reader to wonder if the use of such a phrase as "misty profundity" by any writer is not a bit nonsensical. Again, what choice must the ordinary mind make in deciding between Weierstrass who says "a mathematician who is not also something of a poet will never be a complete mathematician," and Ernst Mach, who assigns "the power of mathematics to its evasion of all unnecessary thought and its wonderful saving of mental opertions."

However, when a final analysis can be had, it may be found that diverse interpretations of the meaning of mathematics by uncounted competent minds of the past and present have arisen from the varying emphasis these minds have placed on different phases and functions of the science. For, truly numerous must be the phases and functions of a science having for its main concern such a perfection of the art of logical inference that its applications are demanded in all fields where thinking must be controlled and accurate.

S. T. Sanders.

Matrices and Quadric Surfaces

By JOHN M. H. OLMSTED
University of Minnesota

Introduction. The fact that there is a considerable lag between the development of new mathematical ideas and their introduction in the class room, particularly at the undergraduate level, is well recognized but nevertheless regrettable. Such notions as *group*, *ring*, and *matrix*, for example, could be presented or treated more fully, in the college curriculum, to the consequent advantage of the student, who too frequently has only a vague impression that a field is a system where he can add and multiply without getting into trouble and that a matrix is some sort of array used in "higher mathematics". Such material could be discussed advantageously, for example, in a course in Theory of Equations, where topics of such far less current importance as Sturm's theorem could be reduced to a remark and a reference to a suitable text. A result of such an early training would be a simplification of many parts of a course in Solid Analytic Geometry. The analysis of a system of planes, for example, is facilitated by use of ranks of matrices. A more striking application is the study of the general equation of the second degree and the classification of quadric surfaces. It is the purpose of this paper to present a discussion of this application.

The paper consists of two parts. The first part is elementary and assumes on the part of the reader only a basic acquaintance with matrices, including such ideas as rank, sum and product of two matrices, unit, inverse, transposed (conjugate), and symmetric matrix, and the fundamental relations between matrices and linear transformations, and matrices and quadratic forms. This material is readily available. (Cf. references 1 and 2 at the end of the paper.) Part two completes the classification of quadric surfaces by means of a study of the characteristic equation of the fourth order matrix associated with such a surface. A necessary and sufficient condition for a quadric surface to be imaginary is also obtained. Although the analysis is confined to quadric surfaces in the Euclidean space of three dimensions, the methods are general and provide a complete classification of quadric surfaces for n dimensional space. A summary of this analysis is given at the end of the paper. These results are believed to be new. For their establishment reducibility of a symmetric matrix to canonical form by means of an orthogonal matrix is used. Throughout the paper all matrices are assumed to be real and square.

The term *invariant* will be used to denote any property which is invariant with respect to a certain type of transformation. Thus an invariant may be a number, such as the rank of a matrix, or simply the algebraic sign of an expression.

Frequent reference is made to Fine and Thompson,[3] which will be abbreviated F. and T. The reader's attention is called to a paper by Bush,[4] in which a complete classification of quadric surfaces in three dimensions is obtained by a different method.

PART I

It is disturbing that, in the case of non-homogeneous coordinates, orthogonal matrices correspond to only certain distance-preserving space transformations, and one is led to ask whether a translation can be treated in a fashion somewhat similar to that which is applicable in the case of orthogonal matrices. In order to discuss this question, and to have them at hand, we give certain definitions.

Definition. *A (non-singular) matrix is orthogonal if and only if its inverse is equal to its transpose:* $C^{-1}=C'$, *or* $CC'=I$.

An immediate consequence of the definition is that the determinant of an orthogonal matrix is ± 1. A linear transformation with a third order orthogonal matrix will be called a *rotation* if the determinant is 1 and a *skew-rotation* if the determinant is -1. In this case the origin is fixed. In general a rotation (skew-rotation) is any transformation which is a combination of a rotation (skew-rotation) with origin fixed and a translation. It can be shown that any distance-preserving transformation in space is a combination of rotations, and translations, and that rotations and translations preserve sense, while skew-rotations reverse sense.

As one simplification introduced by the use of matrices, it is interesting to notice that the 22 relations given in F. and T., page 261, are equivalent to the single statement that the matrix of the transformation is orthogonal.

In order to handle translations by means of matrices we shall introduce an artificial fourth variable (which will not seem so artificial to the reader familiar with homogeneous coordinates). Any point in three-dimensional Euclidean space will correspond to an ordered set of four numbers $(x,y,z,1)$, which we shall write (x_1,x_2,x_3,x_4). Then the translation $x=x'+s_1$, $y=y'+s_2$, $z=z'+s_3$ can be written:

$$x_i=\sum_{j=1}^{4} c_{ij}x_j',$$

where $$C=\|c_{ij}\|=\begin{Vmatrix} 1 & 0 & 0 & s_1 \\ 0 & 1 & 0 & s_2 \\ 0 & 0 & 1 & s_3 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Notice that the matrix corresponding to a translation is *not* in general orthogonal. However, the determinant $|C|$ of C is equal to 1.

With respect to our four variables, a rotation or skew-rotation with origin fixed can be expressed in the same fashion:

$$x_i=\sum_{j=1}^{4} c_{ij}x_j'$$

where $$C=\|c_{ij}\|=\begin{Vmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

In this case the third order matrix c of the c's and the fourth order matrix C are both orthogonal.

The general equation of the second degree in three variables, with real coefficients, assumes a form where the left number is a quadratic form:

$$\sum_{i,j=1}^{4} a_{ij}x_ix_j=0,$$

with a real symmetric matrix A. The homogeneous second degree part is also a quadratic form, with a real symmetric matrix a. We shall denote by R and r the ranks of the matrices A and a, and by Δ and D their determinants.

Let us recall (1) that if the variables of a quadratic form with matrix A are subjected to a non-singular linear transformation with matrix C, the quadratic form is transformed into another quadratic form with symmetric matrix $C'AC$, (2) that the rank of the product of a matrix A and a non-singular matrix C is equal to the rank of A, and (3) that the determinant of the product of two matrices is equal to the product of their determinants. In the case of the general equation of the second degree we can deduce:

(1) A translation leaves unchanged the matrix a, and the rank and determinant, R and Δ, of the matrix A.

(2) A rotation or skew-rotation leaves unchanged the ranks R and r, and the determinants Δ and D.

By combining these, we have the important

Theorem. *Under any distance-preserving transformation in three-dimensional space, R, r, Δ, and D are invariant.*

The *characteristic equation* of the third order matrix a is defined as the equation obtained by setting equal to zero the third order determinant $|a-kI|$, where I is the third order unit matrix and k is the variable. If now the variables x_1, x_2, and x_3 are transformed by a rotation or skew-rotation, with matrix c, the characteristic equation of the third order matrix b of the new quadratic form is $|b-kI|=0$, where $b=c'ac$. We shall show that the new characteristic equation is equal to the original, identically in k. For any number k,

$$|a-kI| = |c'(a-kI)c| = |c'ac-kc'Ic| = |b-kI|.$$

Therefore the coefficients and the *characteristic roots* of the characteristic equation of the third order matrix of a quadratic form are invariant under a rotation or skew-rotation. Since a translation leaves the matrix a unchanged, we have the following theorem (using the notation of F. and T.):

Theorem. *With respect to any orthogonal transformation the following are invariant: R, r, Δ, D, J, I, k_1, k_2, k_3.*

For a simple proof that the characteristic roots of a real symmetric matrix are real the reader is referred to McConnell,[5] page 16.

We shall find it convenient to use the word *congruent* according to the following:

Definition. *Two real symmetric matrices are* congruent *if and only if one can be obtained from the other by a combination of the following* congruence transformations: *rotations, skew-rotations, translations, and multiplication by non-zero real constants; in other words if and only if they correspond to second degree equations whose graphs are congruent.*

Note that congruence is an equivalence relation; that is, it is reflexive, symmetric, and transitive.

It is important to determine what properties of a real symmetric matrix are invariant under the final type of congruence transformation, multiplication by a non-zero real constant p. It is easily seen that R and r are unchanged, that Δ, D, J, and I are multiplied by p^4, p^3, p^2, and p, respectively, and that the charactertistic roots, k_1, k_2, and k_3, are also multiplied by p. Combining these results with the last theorem, we have the

Theorem. *With respect to any congruence transformation the following are invariant: R, r, sign of Δ, and whether the non-zero* k's *have the same sign.*

Using these invariants, we have only to determine their values or signs for each type of second degree equation in simplest form in order to obtain the following analysis, where the surfaces are arranged according to the 16 [actually 17 because type 15 is in fact two] classifications of F. and T., page 245. It may be noted that this analysis is not sufficient to determine whether an elliptic cylinder is real or imaginary (cases 9 and 11), or whether two parallel planes are real or imaginary (case 15). Answers to these questions will be given in the second part of this paper.

Analysis of the General Equation of the Second Degree

R	r	Sign of Δ	k's same sign	Surface	Number
4	3	−	Yes	Ellipsoid (real)	1
4	3	+	Yes	Ellipsoid (imaginary)	2
4	3	+	No	Hyperboloid of one sheet	3
4	3	−	No	Hyperboloid of two sheets	4
3	3		No	Cone (real)	5
3	3		Yes	Cone (imaginary)	6
4	2		Yes	Elliptic paraboloid	7
4	2		No	Hyperbolic paraboloid	8
3	2		Yes	Elliptic cylinder (real)	9
3	2		No	Hyperbolic cylinder	10
3	2		Yes	Elliptic cylinder (imaginary)	11
2	2		No	Intersecting planes (real)	12
2	2		Yes	Intersecting planes (imaginary)	13
3	1			Parabolic cylinder	14
2	1			Parallel planes (real)	15
2	1			Parallel planes (imaginary)	15
1	1			Two coincident planes	16

Before proceeding further, it might be well to note how easily the ranks R and r may be obtained *simultaneously*, by considering a simple example: Classify the surface:

$$x^2+4y^2+9z^2-4xy+6xz-12yz+2x+10y+7=0.$$

The matrix for this equation is

$$\begin{Vmatrix} 1 & -2 & 3 & 1 \\ -2 & 4 & -6 & 5 \\ 3 & -6 & 9 & 0 \\ 1 & 5 & 0 & 7 \end{Vmatrix}.$$

Add twice the first row to the second, subtract three times the first

row from the third, and subtract the first row from the fourth. Then carry through similar operations on the columns. The result is

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -3 \\ 0 & 7 & -3 & 7 \end{vmatrix}.$$

Further simple alterations are possible, but it is already easy to see that $R=3$ and $r=1$. Therefore the surface is a *parabolic cylinder.*

In general, in obtaining the ranks R and r, all the elementary transformations on matrices can be carried through with this important exception: Never add a multiple of the fourth row (column) to any other row (column).

PART II

One might ask what significance the characteristic roots of the fourth order matrix have in the classification of second degree surfaces. The difficulty lies in the fact that the matrix defining a translation is not orthogonal. A simple example is sufficient to show that under a translation, the characteristic roots are *not* invariant.

For simplicity, let us define two sets of characteristic roots to be *similar* if and only if they have the same arrangement of signs, except possibly for a negative factor. For example, roots with signs $(+, +, -, 0)$ are similar to roots with signs $(-, 0, +, -)$. Using the idea of similarity of charactertistic roots, we shall prove the following

Theorem. *If a real symmetric matrix is transformed by a congruence transformation, the characteristic roots of the new matrix are similar to those of the original.*

This theorem is true in spaces of any number of dimensions, but we shall confine ourselves to three dimensional space for simplicity, the methods of proof being the same. From results already obtained, proof is immediately available for a rotation, skew-rotation, or transformation obtained by multiplying all elements of the matrix by a non-zero real constant. Since any translation is a combination of translations in three basic directions, we shall confine ourselves in the proof to a translation given by a matrix C as introduced in Part I, where $s_1=s$, $s_2=s_3=0$.

Define $F(s,k) \equiv |C'AC-kI|$. Then

$$F(s,k)=k^4+P(s)k^3+Q(s)k^2+R(s)k+\Delta,$$

where $P(s)$, $Q(s)$, and $R(s)$ are polynomials in s of degree ≤ 2, and Δ is independent of s.

Lemma 1. *If* $\Delta = R(0) = 0$, *then* $R(s) \equiv 0$.

Proof: There exists an orthogonal matrix T such that $T'AT$ is a diagonal matrix with diagonal elements u_1, u_2, u_3, u_4, where

$$|u_1| \geq |u_2| \geq |u_3| \geq |u_4|.$$

Then $\Delta = u_1u_2u_3u_4$ and $R(0) = -(u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4)$.

The assumptions of the lemma imply that $u_3 = u_4 = 0$. Therefore the rank of A is ≤ 2. Finally, since the coefficients of s^2 and s in $R(s)$ are linear combinations of cofactors of elements of A, it follows that $R(s) \equiv 0$.

Lemma 2. *If* $\Delta = R(0) = Q(0) = 0$, *then* $Q(s) \equiv 0$.

Proof: As above, $u_2 = u_3 = u_4 = 0$. Therefore the rank of A is ≤ 1. Since the coefficients of s^2 and s in $Q(s)$ are linear combinations of second order subdeterminants of A, it follows that $Q(s) \equiv 0$.

Lemma 3. *If* $\Delta = R(0) = Q(0) = P(0) = 0$, *then* $P(s) \equiv 0$.

Proof: In this case the rank of A is zero, and the result follows immediately.

Using the fact that the result of combining two translations corresponding to s and $-s$ is the identity transformation, and applying arguments similar to those used in the three lemmas above, one can establish the following lemmas:

Lemma 4. *If* $\Delta = 0$, *then* $R(s)$ *vanishes identically or never.*

Lemma 5. *If* $\Delta = 0$ *and* $R(s) \equiv 0$, *then* $Q(s)$ *vanishes identically or never.*

Lemma 6. *If* $\Delta = 0$ *and* $R(s) \equiv 0$, *then* $P(s)$ *vanishes identically or never.*

Finally, if s is allowed to vary continuously from zero to any given value, the characteristic roots must vary continuously without changing sign. For, if any root changed sign, it would vanish for some value of s, and the power of k factorable from $F(s,k)$ would change. This would contradict one of the last three lemmas. This completes the proof of the theorem stated at the beginning of Part II.

To complete the classification of quadric surfaces in three dimensions we give the following test for distinguishing between the real and imaginary surfaces of cases 9, 11, and 15, proof being given

immediately by the theorem just established: If the non-zero characteristic roots of A have different signs the surface is real; if the signs are the same the surface is imaginary.

An interesting generalization of this test, true for any number of dimensions, is also a consequence of the theorem above:

Theorem. *A necessary and sufficient condition for a real second degree equation to have an imaginary graph is that its matrix or order* $n+1$ *have rank* >1 *and no characteristic roots of opposite signs.*

Note: Imaginary cones and planes and their higher dimensional analogues are regarded as imaginary surfaces, even though they consist partially of real points.

It has been frequently remarked that certain results apply to spaces of any number of dimensions. This is also true regarding the completeness of the characterization given in this paper. Well-known methods of reduction of second degree functions to standard form complement the first theorem of Part II and enable us to state a final inclusive theorem, applicable to Euclidean space of n dimensions. For this purpose we shall restrict ourselves as before to quadric surfaces which are graphs of real quadratic equations, and define two quadric surfaces to be of the *same type* if and only if they are congruent except for scales chosen on a properly chosen orthogonal set of axes.

General Characterization Theorem. *A necessary and sufficient condition for two quardric surfaces to be of the same type is that their matrices*

$n=1$			$n=3$		
++	+	imaginary points	++++	+++	imaginary ellipsoid
+−	+	real distinct points	+++−	+++	real ellipsoid
+	+	coincident points	+++−	++−	hyperb., 2 sheets
			++−−	++−	hyperb., 1 sheet
			+++−	++	ell. paraboloid
			++−−	+−	hyp. paraboloid
$n=2$			+++	+++	imaginary cone
			++−	++−	real cone
+++	++	imag. ellipse	+++	++	imag. ell. cylinder
++−	++	real ellipse.	++−	++	real ell. cylinder
++−	+−	hyperbola	++−	+−	hyperbolic cylinder
++−	+	parabola	++−	+	parabolic cylinder
++	++	imag. int. lines	++	++	imag. int. planes
+−	+−	real int. lines	+−	+−	real int. planes
++	+	imag. par. lines	++	+	imag. par. planes
+−	+	real par. lines	+−	+	real par. planes
+	+	coincident lines	+	+	coincident planes

of order n have similar characteristic roots and their matrices of order $n+1$ have similar characteristic roots.

Classification of the quadrics on this basis is tabulated below for dimensions 1, 2, and 3, and can easily be obtained for higher dimensional spaces. As seen in the following table, if n is given the values 1, 2, and 3, then the number N of distinct quadrics has the values 3, 9, and 17, respectively. An induction proof can be supplied to validate the general formula $N=n^2+3n-1$. In each case below the first column contains a representative of the similarity class of characteristic roots of the matrix of order $n+1$. and the second a corresponding representative for the matrix of order n, zeros being omitted consistently.

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Equilong Geometry of Third Order Differential Elements[1]

By JOHN DE CICCO
Illinois Institute of Technology

1. *Introduction.* The study of the equilong geometry of analytic arcs or curves was first developed by Kasner about 1912.[5] One of the most interesting features of this is the equilong measure of a horn angle. In this paper, we shall discuss the geometry of a fundamental six-parameter group G_6 which is based essentially on this equilong horn angle measure. This is analogous to the Kasner-Comenetz group G_6, discussed by Comenetz in the paper, *Kasner's invariant and trihornometry*, American Mathematical Monthly, Vol. 45, pp. 82-87, February, 1938.

A third order differential element consists of a line (u,v) and the first three derivatives of v with respect to u at that line. Therefore any third order differential element is defined by the number quintuplet (u,v,v',v'',v'''). The totality of third order elements is ∞^5. In our work, it is found most convenient to define a third order differential element by the number quintuplet (u,v,w,r,s), where (u,v) are the hessian coordinates of the line, $w=v'$, r is the radius of curvature, and s is the rate of variation of the radius of curvature per unit radian measure of the angle that the tangent line makes with a given line at the line (u,v). Since we shall confine ourselves to the third order differential elements at a fixed line, any such element may be given given by the number triplet (w,r,s). Our chief interest will then be the study of the three-dimensional manifold which consists of the ∞^3 third order elements (w,r,s) at a fixed line.

Equilong correspondences are those line transformations of the plane which preserve or reverse the distance between the points of contact of any two curves along their common tangent line. According as the distance is preserved or reversed, such a correspondence is said to be direct or reverse. The direct equilong transformations form an infinite continuous group G, whereas the equilong transformations both direct and reverse form an infinite group G'.

It can be shown that the direct equilong group G (or the mixed equilong group G') induces a continuous six-parameter group G_6 (a mixed six-parameter group G'_6) between the ∞^3 third order differential elements at a fixed line. By setting $w=$constant, a certain three-parameter group G_3 is obtained which has been profusely studied by

Kasner in preceding papers and is essentially the same as the geometry in the Kasner plane introduced in the conformal case. It is our intent to give some of the geometry of our larger group G_6 which is essentially different from the analogous Kasner-Comenetz group G_6 in the conformal case, and indicate how some of the results of G_3 are but special cases of these new results.

The direct group G_6 contains ∞^2 involutions, whereas the reverse six-parameter set contains ∞^3 symmetries and ∞^3 inversions. These three types: involutions, symmetries, and inversions are equilongly distinct. They are all the transformations of period two contained in the mixed group G'_6. Any one of our involutions, symmetries, and inversions really consists of the first three terms of the power series developments of an equilong involution, K symmetry, and D inversion.

The groups generated by involutions, symmetries, and inversions are obtained. Our fundamental result is that the total group generated by all transformations of period two is a certain mixed four-parameter set, any transformation of which can be factored into four or fewer inversions by a symmetry.

Under G_6, three elements possess a fundamental invariant (besides the two distances) which is essentially a generalization of Kasner's equilong horn angle measure. In general, n elements possess $(3n-6)$ invariants of which $(n-1)$ are distances and the remaining $(2n-5)$ are of the new type. Any other invariant must be a function of these $(3n-6)$ invariants.

Finally is studied the differential geometry of a series $r=r(w)$, $s=s(w)$. A series possesses two new fundamental differential invariants which we shall term k_1 the curvature and k_2 the torsion. Two series are equivalent under G_6 if and only if their curvatures and torsions are the same functions of the distance w.

2. *The fundamental group G_6.* If m is any arbitrary function of u, not identically zero and Ψ is any other function of u, it is observed that the direct equilong group may be written in hessian coordinates as

$$(1) \qquad U=\int\frac{1}{m}du, \qquad V=\frac{v}{m}+\Psi.$$

Upon extending this three times, t is found that the direct equilong group induces the six-parameter group G_6 between the third order differential elements at the fixed line (u,v)

$$(2) \qquad W=w+h, \quad R=mr+aw+b, \quad S=m^2s+cw+d,$$

where $$h=-\frac{m'}{m}v+m\Psi', \quad a=-m',$$

(3) $$b=\left(\frac{1}{m}-m+\frac{m'^2}{m}-m''\right)v+(\Psi+mm'\Psi'+m^2\Psi''),$$

$$c=1-m^2+m'^2-2mm'',$$

$$d=\left(-\frac{m'}{m}\{1+m'^2\}+2m\,m''-mm'''\right)v$$

$$+m(1+m'^2+mm'')\Psi'+3m^2m'\Psi''+m^3\Psi'''.$$

The reverse equilong transformations yield the following six-parameter set

(4) $$W=-w+h, \quad R=m\tau+aw+b, \quad S=-m^2s+cw+d.$$

The direct transformations (2) form the continuous group G_6, whereas the transformations (2) and (4) form the mixed group G'_6.

3. *The group generated by involutions.* Any correspondence of (2) which is of period two is called an *involution*. There are ∞^2 involutions given by

(5) $$W=w, \quad R=-\tau+aw+b, \quad S=s.$$

Theorem 1. *The group generated by all involutions is the mixed two-parameter set*

(6) $$W=w, \quad R=\mp\tau+aw+b, \quad S=s.$$

The first is an involution and the second is any transformation which can be factored into two involutions in ∞^2 ways.

Let us note that (5) represents the first three terms of the power series development of an equilong involution.[3]

4. *The group generated by symmetries.* The set (4) contains two types of transformations of period two according as $m=\mp 1$. Any such transformation is called a *symmetry* or an *inversuion* according as $m=-1$ or $m=+1$.

There are ∞^3 symmetries given by

(7) $$W=-w+2\omega, \quad R=-\tau+2\rho, \quad S=-s+2\sigma.$$

This leaves only the element (ω, ρ, σ) invariant.

Theorem 2. *The group generated by all symmetries is the mixed three-parameter set*

(8) $$W=\mp w+h, \quad R=\mp\tau+b, \quad S=\mp s+d.$$

The first is a symmetry and the second is any transformation which can be factored into two symmetries in ∞^3 ways.

The preceding theorem is a special case of a certain general theorem stated by Kasner in one of his preceding papers.

5. *The group generated by inversions.* There are ∞^3 inversions given by

$$W=-w+2\omega, \quad R=r+a(w-\omega), \quad S=-s+2\sigma. \tag{9}$$

The product of two inversions is

$$\begin{aligned} W&=w+2(\omega_1-\omega_2),\\ R&=r+a_2(w-\omega_2)+a_1(-w+2\omega_2-\omega_1),\\ S&=s+2(\sigma_1-\sigma_2). \end{aligned} \tag{10}$$

A transformation which can be factored into two inversions must necessarily be of the form

$$W=w+h, \quad R=r+aw+b, \quad S=s+d, \tag{11}$$

where

$$h=2(\omega_1-\omega_2), \quad a=a_2-a_1,$$

$$b=-a_2\omega_2+a_1(2\omega_2-\omega_1), \quad d=2(\sigma_1-\sigma_2). \tag{12}$$

The preceding equations are equivalent to

$$\begin{aligned} &\omega_1=\omega_2+h/2, \quad a_1=-a+a_2, \quad \sigma_1=\sigma_2+d/2,\\ &b=-a(\omega_2-h/2)-a_2h/2. \end{aligned} \tag{13}$$

Theorem 3. *If $h\neq 0$, or $h=0$ and $a\neq 0$, our transformation* (11) *can be factored into two inversions in ∞^2 ways. If $h=a=b=0$, our transformation can be factored into two inversions in ∞^3 ways.*

By the preceding result, if $h\neq 0$, our transformation (11) can be factored into two inversions. Suppose now our transformation is such that $h=0$. Obviously it can be factored into two transformations T_1 and T_2 of the same form (11) such that h_1 and h_2 are not zero and such that $h_1+h_2=0$. Since T_1 and T_2 are factorable into two inversions, it follows that our transformation (8) can be factored into four inversions.

Finally multiplying (11) by an inversion, the resulting transformation can be factored into one or three inversions. Hence

Theorem 4. *The total group generated by inversions is the four-parameter set*

$$W=\pm w+h, \quad R=r+aw+b, \quad S=\pm s+d. \tag{14}$$

Any transformation of this group can be factored into four or fewer inversions.

Upon multiplying (14) by a symmetry, we obtain the following proposition.

Theorem 5. *The total group generated by all transformations of period two is the mixed four-parameter set*

$$W = \pm w + h, \quad R = (\pm \text{ or } \mp)r + aw + b, \quad S = \pm s + d. \tag{15}$$

There are $4\infty^4$ transformations in this set. Any transformation of this group can be factored into four or fewer inversions by a symmetry.

The preceding theorems are special cases of certain general theorems stated by Kasner and the author.[4]

6. *The condition that the product of three inversions be an inversion.* The product of three inversions.

$$\begin{aligned} W &= -w + 2(\omega_3 - \omega_2 + \omega_1), \\ R &= r + a_3(w - \omega_3) + a_2(-w + 2\omega_3 - \omega_2) + a_1(w - 2\omega_3 + 2\omega - \omega_1) \\ S &= -s + 2(\sigma_3 - \sigma_2 + \sigma_1). \end{aligned} \tag{16}$$

Now if the above correspondence is an inversion, there must exist quantities $(\omega_4, a_4, \sigma_4)$ such that

$$\begin{aligned} \omega_4 &= \omega_3 - \omega_2 + \omega_1, \\ a_4 &= a_3 - a_2 + a_1, \\ a_4\omega_4 &= a_3\omega_3 - a_2(2\omega_3 - \omega_2) - a_1(-2\omega_3 + 2\omega_2 - \omega_1), \\ \sigma_4 &= \sigma_3 - \sigma_2 + \sigma_1. \end{aligned} \tag{17}$$

Eliminating $(\omega_4, a_4, \sigma_4)$ from these, we obtain

Theorem 6. *The necessary and sufficient condition that the product of three inversions be an inversion is*

$$a_1(\omega_2 - \omega_3) + a_2(\omega_3 - \omega_1) + a_3(\omega_1 - \omega_2) = 0. \tag{18}$$

7. *The invariants of n elements.* By equations (2) and (4), we find

Theorem 7. *Let three elements of third order (C_1, C_2, C_3) be given such that r_λ is the radius of curvature and s_λ is the rate of variation of the radius of curvature per unit radian measure of the angle of the tangent line of C_λ, and δ_λ is the distance from the point of contact of C_μ to that of C_ν along the fixed line. The three elements (C_1, C_2, C_3) possess as in-*

variants under the continuous group G_6 the three distances δ_1, δ_2, δ_3 $(\delta_1+\delta_2+\delta_3=0)$ and the new fundamental expression

$$\frac{(\tau_1\delta_1+\tau_2\delta_2+\tau_3\delta_3)^2}{s_1\delta_1+s_2\delta_2+s_3\delta_3}. \tag{19}$$

Under the mixed group G'_6 the absolute values of these quantities are invariant. For three elements (C_1, C_2, C_3), only two of which can possibly be tangent, these are the unique invariants.

Of course, if any two of the elements (C_1, C_2, C_3) are tangent, then (19) should reduce to the Kasner equilong measure of a horn angle. As a matter of fact, it is found that (19) becomes the equilong measure

$$M_{12}=\frac{(\tau_2-\tau_1)^2}{s_2-s_1}, \tag{20}$$

of the horn angle formed by the two tangent curves (C_1, C_2).

Theorem 6. *A trihorn (C_1, C_2, C_3), three elements all tangent to each other, possess the three equilong measures M_{12}, M_{23}, M_{31} as the unique invariants.*[6]

The above two theorems demonstrate that three elements (no two or all three tangents) *always* possess three fundamental invariants. By further consideration of our group G_6, it is evident that the following three propositions are true.

Theorem 7. *Under the group G_6, $n\geqq 3$ elements, of which no two are tangent, possess $(3n-6)$ invariants, of which $(n-1)$ are distances and the remaining $(2n-5)$ are of the form* (19). *Any other invariant must be a function of these $(3n-6)$ invariants.*

Theorem 8. *An n-horn (n elements all tangent to each other) possesses $(2n-3)$ independent invariants—the equilong measures of the horn angles. Any other invariant must be a function of these $(2n-3)$ measures.*

Theorem 9. *Let n elements be such that τ_1 form an τ_1-horn, τ_2 form an τ_2-horn, $\cdots$, τ_k form an τ_k-horn where $\tau_1+\tau_2+\cdots+\tau_k=s$, and the remaining curves are not tangent to each other nor to the different τ_λ-horns. For these n elements, we have the following types of independent invariants:* (1) $(n-s+k-1)$ *distances,* (2) $(2s-3k)$ *equilong measures, and* (3) $(2n-2s+2k-5)$ *invariants of type* (19). *Any other invariant must be a function of these $(3n-s-6)$ invariants.*

This completes the discussion for the invariants of n elements. In the following section, we shall consider two important fundamental differential invariants.

8. *The curvature and torsion of a series of elements.* By differentiating (2) with respect to w, we find

$$(21) \qquad \frac{d^2R}{dW^2} = m\frac{d^2\tau}{dw^2}, \qquad \frac{d^3R}{dW^3} = m\frac{d^3\tau}{dw^3},$$

$$\frac{d^2S}{dW^2} = m^2\frac{d^2s}{dw^2}.$$

Upon eliminating m from these, we find

Theorem 10. *A series $\tau = \tau(w)$, $s = s(w)$ possesses the two differential invariants*

$$(22) \qquad K_1 = \frac{d^3\tau/dw^3}{d^2\tau/dw^2}, \qquad K_2 = \frac{d^2s/dw^2}{(d^2\tau/dw^2)^2}.$$

We shall call K_1 the curvature and K_2 the torsion of our series.

Now let two series $\tau = \tau(w)$, $s = s(w)$, and $R = R(W)$, $S = S(W)$ be such that their curvatures and torsions are the same functions of the distance w. By integrating (22), we obtain (21). Since $dW = dw$, we easily deduce from this and (21) the equations (2).

Theorem 11. *Two series are equivalent under the group G_6 if their curvatures K_1 and torsions K_2 are the same functions of the distance w. Any two arbitrary functions of w may be taken to represen the curvature and torsion of a series. There are then ∞^6 such series all equivalent under G_6.*[7]

In a later paper, we shall discuss the differential geometry of a congruence (∞^2) of elements.

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A Mathematical Support for the Theory of the Conservation of Energy

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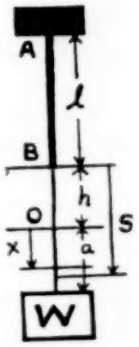
Suppose a rubber band, of natural length l, to be suspended vertically from a fixed point and a weight w to be attached to the lower end. Let the weight stretch the band to the length $l+h$. If the weight is given a displacement $a<h$ and then released the equation of motion of the weight may be found as follows

The force on the rubber band will be $f=k(h+x)$ and becomes w when $x=0$. Therefore $k=w/h$, and the differential equation of the motion is then



$$(1) \qquad \frac{w}{g}\,\frac{d^2x}{dt^2}=w-\frac{w}{h}(h+x),$$

with the initial conditions

$$t=0,\ x=a,\ \frac{dx}{dt}=0.$$

This equation is readily solved to find that the law of motion of the weight is

$$(2) \qquad x=c_1\sin t\sqrt{g/h}+c_2\cos t\sqrt{g/h},$$

and the initial conditions show that $c_2=a$, and $c_1=0$.

Now suppose the weight is let fall from a height b above B, the lower end of the unstretched band. How far will it fall?

This makes the weight act as a falling body for $-b-h\leqq x\leqq -h$, and hence has the differential equation

$$\frac{w}{g}\,\frac{d^2x}{dt^2}=w,$$

with initial conditions

$$x=-b-h,\ \frac{dx}{dt}=0,\ t=0.$$

The solution of this is $x=\frac{1}{2}gt^2-b-h$, where t, determined by $b=\frac{1}{2}gt^2$, is $\sqrt{2b/g}$. Then at $x=-h$ we have

$$\frac{dx}{dt}=\sqrt{2-bg},$$

along with $t=\sqrt{2b/g}$.

These derived values of x, dx/dt, and t may be considered as initial conditions for equation (1), since the weight follows that law, if $x \geqq -h$. Using these in (2) and its derived equations for dx/dt and d^2x/dt^2, we have the law of motion of the weight for $x \geqq -h$, as

$$\begin{aligned}(3)\qquad x=&(\sqrt{2bh}\cos\sqrt{2b/h}-h\sin\sqrt{2b/h}\sin t\sqrt{g/h}\\&-(h\cos\sqrt{2b/h}+\sqrt{2bh}\sin\sqrt{2b/h}\cos t\sqrt{g/h}.\end{aligned}$$

Now the amplitude of this sine function will be the maximum value of x and it is

$$\sqrt{(\sqrt{2bh}\cos\sqrt{2b/h}-h\sin\sqrt{2b/h})^2+(\sqrt{2bh}\sin\sqrt{2b/h}+h\cos\sqrt{2b/h})^2},$$

or $\sqrt{2bh+h^2}$. Therefore the lowest point to which the weight will drop will be $h+\sqrt{h^2+2bh}$ below B.

Now consider this second part of the problem as an illustration of the theory of the Conservation of Energy. That is, that the change in potential energy of the weight as it falls from $x=-h-b$, the position of the weight when dropped, will equal the kinetic energy of the weight while in motion or the work done during the fall when it has reached the bottom of its fall.

This would give, considering the work done,

$$(4)\qquad w(b+h+x)=\int_0^{x+h}\frac{w}{h}\,s\,ds,$$

where s is measured from B. Hence

$$b+h+x=(x+h)^2/2h,$$

where x is the distance the weight will fall below the equilibrium point O.

Solving this, we have

$$x=\sqrt{2bh+h^2},$$

and hence the complete fall below B is

$$h+\sqrt{2bh+h^2},$$

the same as shown by the mathematical solution above.

We could also use the energy integral derived from the weights two differential equations

$$\frac{w}{g}\ \frac{d^2x}{dt^2}=w,\ -b-h\leqq x\leqq -h, \tag{5}$$

and

$$\frac{w}{g}\ \frac{d^2x}{dt^2}=-\frac{w}{h}\,x,\ x\geqq -h. \tag{6}$$

Thus, using $v=dx/dt$ as an integrating factor for (5) and (6), we have after integration of left member,

$$\tfrac{1}{2}\,\frac{w}{g}\,v^2\Big]_0^v=\frac{w}{g}\int_{-b}^{-h}\frac{dx}{-h}-\frac{w}{h}\int_{-h}^{x}x\,dx,\ x\geqq -h. \tag{7}$$

This becomes $$\tfrac{1}{2}mv^2=\frac{mg}{2h}\,(2bh+h^2-x^2),$$

and hence the weight would stop at $x=\sqrt{2bh+h^2}$. The complete fal below B is therefore the same as found from the work integral.

Hence the mathematical solution and the Conservation of Energy Theory agree.

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Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

A Tenth Lesson in the History of Mathematics

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19. *Theory of Groups.* Although the theory of groups extends into various fields of advanced mathematics and is commonly taught in the universities as an advanced branch thereof, its principles appear also in some of the most elementary parts of our subjects. Hence it may be useful to give here a brief introduction to the history of this subject even if it will not be possible to note therein many of most inspiring advances which were due to an emphasis on the ideas that are involved in this comparatively modern subject. While it is true that the history of a subject cannot be fully comprehended without a mastery of the subject itself, much mathematical work has been inspired by an effort to clarify ideas which appeared at first in a vague form. Vague fruitful ideas are often more valuable than clear ideas along less fruitful lines.

We begin with several examples of groups without giving a definition of this term in the hope that these examples will make some of the ideas of a group clear. The totality of the integers together with zero constitute a group when these integers and zero are combined by addition. The number zero is the identity of this group and each of the negative integers is the inverse of the corresponding positive integer, and vice versa. Every group contains one and only one identity element, and this is its own inverse. A necessary and sufficient condition that a given element be its own inverse is that its square is the identity. Hence the given group contains only one element that is its own inverse, namely, zero.

Each of the other elements of this group is of infinite order in the sense that it can be repeated an endless number of times without arriving at the identity. While this group is old it did not explicitly exist before zero and negative numbers were regarded as numbers

and hence it has a long history and was not a clearly recognized part of mathematics until about the beginning of the nineteenth century. It should, however, be noted that negative numbers and zero were often used correctly long before this time. This group formed by the positive and negative integers together with zero is said to be of infinite order since it involves an infinite number of elements; the order of a group being the number of the distinct elements thereof, and no two elements of a group are equal to each other.

Examples of groups of finite order with respect to addition are furnished by reducing the sums that are equal to or exceed a given positive modulus with respect to this modulus when the integers which are less than this modulus are combined by addition. The order of this group is equal to this modulus, and such groups have been employed since very early times, as, for instance, in the days of a week. Such modular groups are among the oldest finite groups in the mathematical literature. They are cylic, and hence each of them is generated by at least one of the numbers contained therein. A necessary and sufficient condition that each of the non-zero elements of such a group is a generator of the group when the order exceeds unity is that the modulus is a prime number.

The concept of cyclic group is one of the oldest concepts of all mathematics and appears in geometry as well as in arithmetic. For instance when a regular polygon is rotated into itself around its center the rotations constitute a cyclic group whose order is equal to the number of the vertices of this regular polygon. Just as the concept of regular polygon cannot be traced to its origin, so the concept of cyclic group cannot now be traced to its origin, and hence the history of group theory, as well as the histories of many other parts of mathematics, must be confined to the later developments of the subject. This means that the history of mathematics is necessarily confined to only a part of the developments relating to our subject, and this part seldom extends to the beginnings.

The totality of the positive rational numbers with the exception of zero constitutes another very important ancient group when these numbers are combined by multiplication. This group is of infinite order, and unity is the identity thereof, while the inverse of an element of this group is the reciprocal of the number representing this element. A simple multiplication group consists of the powers of a given number greater than unity. In the eighth of these lessons it was noted that the ancient Babylonians used the same symbol for all the different numbers of the group composed of the integral powers of 60. The fact that the single number zero has to be excluded if we consider a group formed by the multiplication of ordinary numbers is due to the

exceptional property that the product of zero and any finite number is zero. Group elements should always be assumed to satisfy the condition that whenever any two of the three elements x,y,z belong to a group, then the equation $xy=z$ must uniquely determine the third of these elements as an element of the same group. This is a very useful part of the definition of a group, irrespective of the order of the group. Cf. Weber-Kleine's *Lehrbuch der Algebra*, page 180 (1812).

A very important class of groups of finite order is composed of the positive integers which are less than a given modulus and prime to this modulus when these numbers are combined by multiplication and the products are reduced with respect to this modulus whenever they exceed this modulus. These groups are not necessarily cyclic and hence they furnish more interesting examples of groups than those resulting from addition with respect to a given modulus. For example, when the modulus is 8 there results the non-cyclic group of order 4 which has received a number of other names including those of axial group and the four group. When the modulus is 5, there resu ts the cyclic of order 4. There are two and only two abstract groups of order 4. This very elementary fact seems to have been first explicitly noted by the well-well-known English mathematician A. Cayley in the *Philosophical Magazine*, Volume 7, page 40 (1854). It was known earlier that there are at least two such groups. For instance, J. L. Lagrange, (1736-1813) knew it.

In a mathematical group of finite order it results from the property that the product of every pair of elements is contained in the group that the product of every pair of elements is contained in it, but this is not true as regards groups of infinite order. This fact gives rise to the question whether there are groups in which are elements whose inverses do not appear in the group. On page 163, Volume 1 (1888), of the work entitled *Theorie der Transformationgruppen*, a noted Norwegian mathematician, Sophus Lie (1842-1899), said that the transformations $x'=ax$, where the absolute value of a is less than unity, constitute a group which contains neither the identity nor the inverse of any one of its elements. It is clear that the same conclusion might have been reached for similar reasons by assuming that the absolute value of the number a in the given transformations is always greater than unity but not infinite.

If it is assumed that a may be equal to unity these transformations again include the product of every two of them without including the inverse of any one of them besides the identity; which may also appear in the set, and is then its own inverse. As evidence of the inadequate notion of group on the part of some of the most noted mathematicians and workers in group theory at that time, it may be noted that about

five years after Sophus Lie made the given assertion, Felix Klein stated, on page 66 of Volume 43 (1893) of the *Mathematische Annalen*, that Sophus Lie was the first to point out that for groups of infinite order it is not a consequence of the group concept that the inverse of every element of a group appears in the group. He acknowledged here that in his well-known *Erlanger Program* (1872) he tacitly made this assumption but that it should have been explicitly stated as a property of the particular groups which he then considered.

It is questionable whether any other publication contributed more than Felix Klein's *Erlanger Program* towards making the subject of groups of infinite order widely known and highly appreciated, especially in geometric investigations. Felix Klein regarded therein all of geometry as a group theory problem. This *Program* was translated into Italian (1890), into French (1891), into English (1893), into Polish (1895), into Russian (1896), and into Hungarian (1897). These translations in rapid succession, exhibit the rapid growth of an interest in the group concept at that time, especially as regards groups of infinite order.

About the time of this rapid increase of an interest in group theory the University of Chicago opened its doors to students (1892) and its first Annual Register (1892-1893) announced two courses in group theory, namely, one by O. Bolza (1857-1942), entitled "Theory of Substitutions and its applications to algebraic equations", and the other by H. Maschke (1853-1908) entitled "Finite groups of linear substitutions". The following Annual Register of this university announced an additional course in this subject by E. H. Moore (1862-1932) entitled "Groups". The rapid increase of interest in group theory at the time when the University of Chicago opened its doors to students is also illustrated by the fact that at the mathematical congress held at this university in connection with the World's Columbian Exposition (1893) there were five sessions for the consideration of papers and one of these was devoted to the subject of groups. This fact tended to make group theory widely known in the United States.

To understand the later history of group theory it is necessary to bear in mind that many writers used the term group somewhat vaguely. This was partly due to the fact that permutation groups were the first groups to receive much attention. When permutations are combined the fundamental associative law is satisfied and hence it is easily seen that a finite set of distinct permutations constitutes a group if it satisfies the condition that the product of every two of these permutations and the square of every one of them is contained in the set. It is however desirable to say that every set of distinct operators which satisfies the condition that it includes the product

of every two of them and the square of every one of them is a group. Hence it results that when the subject of abstract groups began to be studied it became necessary to formulate accurate definitions of abstract groups, and this was done for the first time in a satisfactory manner by H. Weber (1842-1913) in 1893, although some steps in this direction had been taken somewhat earlier.

Both in the development of the groups of finite order and in the development of groups of infinite order the applications at first attracted the widest attention. In the former case it was the applications to the solutions of algebraic equations which first attracted wide attention to the work of E. Galois (1811-1832) and others. In the latter case the applications to geometry and the solution of differential equations first attracted wide attention to the group theory work of Felix Klein (1849-1925) and Sophus Lie (1842-1899). The theory of continuous groups is very largely due to the latter of these prominent writers, whose *Collected Works* appeared in six volumes and was published in Germany by B. G. Teubner. It is largely due to the fact that the applications were the main objective of most of the early writers that the term group was at first used with a wider meaning than is usually given to it in abstract mathematics. It should be emphasized that the meaning of the term group is a matter of definition and one should insist only that it be used in accord with a well understood definition.

While Sophus Lie, Felix Klein and Henri Poincaré (1854-1912) were probably the most influential in making the theory of groups none of them interested himself much in the study of the fundamental laws which underlie the group concept and which had to be clearly formulated for the purpose of a logical presentation of its elements. Arthur Cayley (1821-1895), Leopold Kronecker (1823-1892), and Heinrich Weber (1842-1913), on the contrary, interested themselves with success in the latter study and in emphasizing the importance of the associate law as a fundamental element of the group concept. The importance of this law was also emphasized by the Irish mathematician W. R. Hamilton (1805-1865) who is widely known for his discovery of quaternions in 1843 and who also contributed to the early development of the abstract theory of groups along fruitful lines.

In particular, W. R. Hamilton was the first to direct attention to the abstract definitions of the groups of the regular solids. He noted that the regular tetrahedron admits two movements of order 3 whose product is of order 2. If we repeat the movements by s_1 and s_2 respectively, then the group generated by s_1 and s_2 can easily be proved to be of order 12 and to be simply isomorphic with the alternating group of degree 4. Similarly, if s_1 and s_2 represent two operations

of orders 3 and 4 respectively whose product is of order 2, then the group generated by s_1 and s_2 is of order 24 and is simply isomorphic with the symmetric group of degree 4 or with the group of movements of the cube. W. R. Hamilton was especially interested in the group generated by an operator of order 3 and an operator of order 5 whose product is of order 2. This group is the well-known simple group of order 60, which is simply isomorphic with the alternating group of degree 5.

He called this group the icosian calculus and said that it furnished "a system of non-commutative roots of unity which are entirely distinct from the i,j,k of the quaternion though having some analogy thereto". The movements of the regular solid furnish one of the most interesting systems of groups of finite order. They have been generalized along various lines by the present writer and some of these generalizations are treated in the "Finite Groups" by Miller, Blichfeldt and Dickson (1916), which was reprinted with corrections by G. E. Stechert & Co. (1938). In L. Pontrjagin's ***Topological Groups***, page 3 (1939), it is said that "the theory of abstract groups investigates an algebraic operation in its purest form". This important work was translated from the Russian by Emma Lehmer and was published by the Princeton University Press (1939).

The study of the history of such a comparatively modern subject of mathematics group theory may be useful to the student of the history of mathematics as a whole, because it enables him to understand better the nature of some of the d fficulties involved in this general history, since the literature relating to the modern subjects is more extensive than that relating to most of the older subjects. For instance, it may at first seem strange that all the abstract groups of certain low orders were determined before a generally accepted definition of the term group existed. Just as complex numbers were often used correctly long before the theory of these numbers had been fully developed, so the term group was often used correctly before a satisfactory definition of this term had been published. The meaning of mathematical terms at a particular time may differ somewhat from its generally accepted meaning at some other time, and this naturally affects mathematical history in a fundamental way.

The term group was used with a non-technical meaning long before it was employed as a mathematical term. It is often difficult to decide in the early literature on group theory whether a writer used the term with a mathematical meaning or with its then common meaning. This is true, in particular, as regards E. Galois (1811-1832), who is commonly credited for introducing this term as a mathematical term. A. L. Cauchy (1789-1857) used the term "***systeme de substi-***

tutions conjugees" for the same concept. The final adoption of the term group with its modern mathematical meaning was largely due to C. Jordan, who used it frequently in his influential *Traite des Substitutions* (1870) and elsewhere. It should however be noted that C. Jordan used the term only in connection with the permutation groups and not with its present abstract meaning, which is due to H. Weber and others.

In view of the many brief modern mathematical textbooks for beginners in a field of study it may be interesting to note that the work of C. Jordan, which was the first textbook on group theory, was somewhat extensive (xviii+667 large pages) in view of the comparatively limited knowledge in the field at this time. The great richness of the material in this textbook had a great influence in attracting attention to the importance of the subject. The beginner in a field is often more inspired by the wide outlook which it opens than by the simplicity of its elements when these are presented in an attractive form by avoiding difficulties.

Few other subjects have such wide contacts with various branches of mathematics as the theory of groups, and yet involve so much that is easily understood by those who have mastered only small portions of the entire field of mathematics. Fortunately, C. Jordan in his first textbook on the subject was able to exhibit its wide bearings, and thus he attracted wide interest in it even if he failed to discover some of its most useful theorems, such as Sylow's theorem, which was then then within easy reach. The first separate book on group theory published in America was a translation of a German textbook by Eugen Netto (1846-1919). This translation was due to F. N. Cole (1861-1926) and was published under the title *The Theory of Substitutions and its Applications to Algebra* (1892). It was much more elementary than the earlier work by C. Jordan, which was never translated but was more influential later. J. J. Sylvester (1814-1897), who has sometimes been called the founder of the first school of advanced mathematics in America, was the first to give a course in groups in our country. This was given during the first half-year of (1882-1883) at Johns Hopkins university. In European universities such courses had been given much earlier, but America was then far behind some the European countries along mathematical lines. J. J. Sylvester never did much towards the advancement of the subject of group theory. A. Cayley to whom we referred above, did much more along this line, but made many mistakes.

One of the simplest illustrations of a group of finite order is furnished by the eight movements of the plane which transform a rigid square of this plane into itself. Since this group is widely used in the textbooks it may be desirable to note here a slight oversight relating to it which appears on page 152 of the *Survey of Modern Algebra*, by

Birkhoff and MacLane (1941). It is stated here that this group has no outer automorphism. On the contrary, it is easy to see that it has four outer automorphisms as well as four inner automorphisms and that its total number of automorphisms constitute the group which is simply isomorphic with the Sylow subgroup of order 8 contained in the symmetric group of degree 4.

A more extensive account of the early history of group theory than the one given above may be found in my *Collected Works*, 1935 and 1938, respectively. A large number of references relating mostly to the later developments of various parts of the subject appear in the second edition of Volume 1 of the *Encyklopadie der Mathematikhen Wissenschaften*, the publication of which was begun in 1939 but was interrupted by the second World War. The part relating directly to group theory appeared in 1939 but many of the applications of groups appear in other parts of this useful mathematical encyclopedia. In this second edition more than twice as much space is devoted to the subjects of group theory than in the corresponding parts of the first edition.

Group theory was one of the first subjects of advanced mathematics which was vigorously cultivated by American mathematicians and hence its history is of special interest to the American students of the history of our subject. In his retiring address as President of the London Mathematical Society, published in the *Proceedings of the London Mathematical Society*, Volume 7, pages 1-7 (1908), W. Burnside directed attention to the fact that the theory of groups of finite order then aroused the interest of a much smaller number of students in England than on the Continent of Europe and in America. A somewhat unique feature of the history of group theory is the large number of enthusiastic statements relating to the merits and bearing of this subject. A few of those were collected and appear in an article entitled "Appreciative Remarks on the theory of groups", *American Mathematical Monthly*, Volume 10, pages 87-89 (1903).

It should, however, be emphasized that many of the enthusiastic statements relating to group theory are due to men who used the term group with a wider meaning than the one commonly given to this term in treatises on the subject. Hence these statements often fail to exhibit the nature of the subject. It thus happened that too great an importance was assigned to this subject on the part of many who failed to realize the sense in which some of the enthusiastic statements were to be interpreted in view of the conflicting definitions of the term group. In the history of mathematics it is extremely important to note the widely different meanings which have often been assigned to the same technical terms by different writers. The removal of exaggeration is one of the most useful tasks of the mathematical historian.

The Teachers' Department

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Historical Stages in the Definition of Curves

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There is one geometrical element which is perhaps as old as mathematics itself. Rays and segments of straight lines must very early have arisen in the quasi-scientific thinking of early man. A falling stone, the path of light, the distance between villages,—these all suggested the concept which Plato and Euclid later attempted to describe as "lying evenly between its ends", or as "having length but not breadth."

But the very idea of straightness implies the existence of things which do not exhibit this property but are instead curvilinear. The sun and moon, among other things, presented a complementary property of roundness. These two forms—the straight and the round—possessed for the ancients a peculiar fascination, and upon them the Greeks sought to build their astronomy and mechanics, as well as most of their mathematics. The heavenly bodies perforce travelled circularly, in homocentric spheres or in cycles, epicycles, and excentrics. Terrestrial dynamics was compounded of two types of motion—rectilinear and circular. Aristotle explained the law of the lever in terms of circular motion; Archimedes justified it in terms of the bilateral symmetry of the straight line. It was almost two thousand years before other curves began to complement the line and circle in scientific applications in mathematics.

As in Greek science, so also in classical Hellenic geometry there was a similar, albeit less intransigeant, fetichism. The Platonic apotheosis of the straight-edge and compasses, canonized in the postulates of Euclid, has played an enormous role in the history of mathematics. The three famous problems of antiquity were to be solved through the use of lines and circles only; but, fortunately for the development of mathematics, all three of the classical problems are

unsolvable under this restriction. Here one sees a remarkable instance of the fruitfulness of failures, for, defeated in all frontal assaults on the problems, geometers sought partial satisfaction in a disingenuous infraction of the rules. They invented new curves to aid them. These curved lines were derived through two new and distinct modes of definition, both unorthodox in static plane geometry. Some curves were defined kinematically by means of a point subjected to a double motion; others were defined stereometrically through plane sections of familiar solid figures.

We are told that it was Hippias the Sophist who about 420 B.C. invented the first new curve through the intrusion into geometry of the "alien" notion of a mechanical movement. If a horizontal bar or line-segment AB moves downward with a uniform motion of translation to the position OC in the same time that an equal vertical bar or segment OA rotates about O to the position OC, the intersection P of these bars or line-segments will trace out a curve known as the quadratrix of Hippias (see Fig. 1). This curve was used by the Greeks to resolve two of the three classical problems. It easily "solved" all

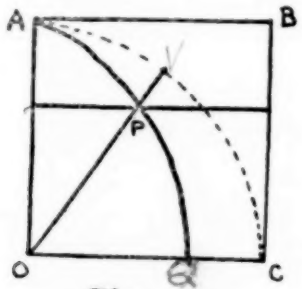

Fig. 1

multisection questions, including that of trisection. One had only to divide the line OA (see Fig. 1) in the desired ratio and find the point P on the curve corresponding to this position of AB. The angle AOP will then give the required section. Moreover, it was shown by Dinostratus that once the quadratrix is constructed, the circle can be squared through the fact that the side of the square AB is the mean proportional between the length of the quarter circle AV and the segment OQ.

The plotting of the curve, however, presents practical difficulties. So long as we have no apparatus for describing the curve by continuous motion, a point-wise construction is necessary even though the language of the definition is kinematic. The distinction between curves defined geometrically and those described mechanically by a continuous motion was not emphasized and we do not know which

point of view Hippias adopted; nor is it known whether he and Dinostratus regarded the curve as furnishing solutions of the classical problems in a strict theoretical sense.

The first "solution" of the Delian problem was carried out along quite different lines. Hippocrates of Chios had made some progress toward the duplication of the cube in showing that if two mean proportionals x and y can be determined so as to satisfy the continued proportion $a : x = x : y = y : 2a$, then the proportional x will be the side of the cube desired, viz. $x^3 = 2a^3$. The problem thus called for the construction, through geometric means, of such a proportional. The first one to cut the Gordian knot in this case seems to have been the Pythagorean scholar Archytas. He is reputed to have determined the required mean proportional through a remarkable construction calling for the intersection of three surfaces of revolution: a cone, a cylinder, and a tore.

His construction is now easily verified by analytic methods: letting the equations of the three surfaces be

$$b^2(x^2+y^2+z^2) = a^2x^2$$

$$x^2+y^2 = ax$$

$$x^2+y^2+z^2 = a\sqrt{x^2+y^2}$$

it is a simple matter to arrange these equations in the form of the continued proportion

$$\frac{a}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{b} \quad . \quad \text{For } b = 2a$$

these equations obviously lead to the solution of the Delian problem. But such an anachronistic application of modern analysis fails to do justice to the ingeniousness of Archytas in inventing this solution with the aid only of synthetic solid geometry. In his day surfaces were not defined by means of equations but by the revolution of known curves such as the line and circle.

The achievement of Archytas in solid geometry paved the way for one of the longest strides in the development of plane curves. Had Archytas studied carefully the sections of his cylinder by a plane, he would have discovered a new curve with striking properties. The ellipse may indeed first have entered Greek geometry as a section of such a cylinder. It may, in fact, have been known to Democritus even earlier, but the evidence here is not convincing. It is reported by Proclus and Eutocius that the ellipse, hyperbola, and parabola all were discovered by Menæchmus toward the middle of the fourth

century B.C. He seems to have been led to them, nevertheless, by following the path which Archytas had suggested—that is, he sought to solve the Delian problem by a consideration of sections of geometrical solids. Taking three right circular cones, one acute-angled, one right-angled, and one obtuse-angled, he cut these by planes perpendicular to an element. (Later Apollonius obtained the three types of curves by varying the angle of the cutting plane with respect to a single fixed cone, rather than by varying the generating angle of the cone.) This disclosed to Greek geometers for the first time a whole family of curves which, unlike lines and circles, differ in shape as well as in size. By means of these conic sections the cube was easily duplicated, either through the determination of the intersection of the two parabolas $x^2 = y$ and $y^2 = 2ax$, or through the intersection of the first of these with the hyperbola $xy = 2a^2$.

The three distinctively different curves which Menæchmus had obtained were destined to play a far larger role in the development of mathematics than their inventor could have anticipated. The classical Greek geometers divided curves into three ranks or orders: the highest place was reserved for the only perfect curves, the line and the circle. These were called the plane loci. Second place was granted to the Menæchmian curves which, probably on account of their original mode of definition, were known as solid loci. All other curves were grouped together under the heading linear loci. Pappus described this last category as made up of those curves "the origin of which is more complicated and less natural [than that of the plane and solid loci], as they are generated from more irregular surfaces and intricate movements." In this description we see the two types of curve definition which the Greeks recognized—the kinematic and the stereometric.

The most important of the Greek linear loci was perhaps the spiral of Archimedes; the locus of a point which moves at a uniform rate along a straight line which revolves about one point on it at a uniform rate in a plane. Other curves defined kinematically were the cissoid of Diocles and the conchoid of Nichomedes. These three curves served to furnish further solutions of the three famous problems.

The most elaborate ancient attempt to extend the number of curves through the stereometric means of definition was made by Perseus, probably during the period between Euclid and Apollonius. He first determined solid figures, called spires, generated by a circle revolving about an axis such that the plane of the circle passes through the axis. There are three general cases, according as the radius of the circle is less than, equal to, or greater than the distance from the

center of the circle to the axis of rotation. Perseus then cut these surfaces by planes, obtaining certain oval-shaped curves (of which the lemniscate of Bernoulli is a special case) which he called spiric sections. He appears to have studied these in a manner comparable to the treatment of conic sections by Apollonius. This type of study contained elements of analytic geometry, but the Greeks never saw in it the possibility of defining an endless variety of other curves.

There are evidences that a few further curves, not all plane, were known to the Greeks. Apollonius may have squared the circle through the cylindrical helix, a curve known later to Geminus, Pappus, and Proclus. Eudoxus had apparently sought to represent the motions of the planets by what was known as the hippopede, a curve perhaps derived through Archytas as the intersection of a sphere with a cylinder. A little later, however, the motions of the planets were represented in the classic manner by Appollonius and others as combinations of circular motions. Such representations are tantamount to constructions of epicyclic curves, but the interest of Greek astronomers and geometers seems to have been focussed on the uniform circular motions rather than on the geometrical curves resulting from these superimposed movements. Even the cycloid appears to have escaped the attention of ancient geometers, although this may have been the "line of double motion" which Iamblichus tells us Karpis of Antioch constructed in order to square the circle.

A comparison of the Greek definition and study of curves with the flexibility and extent of the modern treatment shows surprising narrowness of the ancient point of view. Inspired by the Pythagoreans, they had found number everywhere in nature, but they overlooked much of the geometric beauty which natural phenomena afford. Aesthetically one of the most gifted people of all times, the only curves which they found in the heavens and on the earth were circles and straight lines. Even in theoretical geometry they restricted themselves for the most part to the dyads of Plato and the triads of Menæchmus. The Greeks did not effectively exploit even the two means of definition which they possessed. The kinematic approach and the method of the section of surfaces are capable of far-reaching generalizations, yet scarcely a dozen curves were familiar to the ancients. The failure of ancient mathematicians to develop an analytic geometry and a calculus may well have resulted from the lack of a theory of curves. General algorithms are not necessary where problems concern always one of a limited number of particular curves.

The medieval age is conveniently divided, with respect to the history of mathematics, into an earlier and a later period. The first part covers the long interval between Bœthius and Fibonacci. During

this time interest in mathematics was found chiefly in the Hindu. Arabic, and Byzantine civilizations. Among the Hindus, however, the computational side of the subject overshadowed the logical and speculative aspects, so that one finds there hardly a trace even of the conic sections. Arabic and Byzantine mathematicians took over much of Greek geometry, including that of conics and other curves, but they composed commentaries which did not venture far beyond trodden paths. One therefore finds during the earlier medieval period no striking new development in the theory of curves.

The later portion of the medieval age was characterized by a vigorous intellectual development in the Latin world. In mathematics this manifested itself during the thirteenth century in revived interest in the arithmetic and algebra of Diophantus and al-Khwarizmi, and in the geometry of Euclid and Archimedes. We are told that Campanus and Jordanus Nemorarius generalized the Greek kinematic definition of the conchoid of the line to obtain the limaçon, or the conchoid of the circle, rediscovered four hundred years later by Étienne Pascal. Nevertheless, there was during the succeeding three centuries, no sustained development of these subjectss. The period is more significant for certain abortive efforts in new directions than for solid achievement in the traditional subject-matter. This is especially true in connection with the study of curves. The Greeks had built up an elaborate mathematical theory, but they had applied only the most elementary portions of it to science; the scholastic philosophers of the fourteenth century on the other hand, possessed the most elementary mathematical tools but sought ambitiously to make an elaborate quantitative study of science. Archimedes had never tackled the simple problems of dynamics, even though he possessed rudiments of the calculus; but the scholastics, in what they called the *latitude of forms*, essayed a broad study of physical variables. The word form referred to any quality which admits of variation, and the latitude of a form was the degree to which it possessed this quality. In general the discussion centered about the *intensio* and *remissio* of the form, or the rate of change of the quality. Aristotle had distinguished between uniform and non-uniform velocity, but the scholastics at Oxford and Paris went further. They applied these ideas to acceleration, intensity of illumination or thermal content, and density; and they distinguished not only between uniform and non-uniform rates of change, but subdivided the latter according as the rate of change of *the rate of change* was constant or not. During the second half of the fourteenth century these primitive ideas were given graphical representation by Oresme: the time or duration of a given velocity, for example, was represented by a horizontal line or

"longitude"; and the degree or intensity of the velocity for any time was given by a line perpendicular to it, a "latitude." We see here all the elements necessary for a new and powerful means of defining an endless variety of curves. Oresme's method made possible the pictorial representation of any type of variation of one variable in terms of another. That is, curves might have been defined by analytic means or by the equivalent of differential equations, for this work adumbrated the very important notions of variable, function, and derivative. However, the weakness of medieval mathematical technique precluded the construction at that time of an effective analytic geometry or calculus, and hence this brilliant method of defining curves remained sterile for almost three hundred years. Forms with *latitude uniformiter difformis* (that is, a constant rate of change) were indeed systematically graphed and studied in connection with the law of falling bodies,

$$\frac{v'}{v_2} = \frac{t_v}{t_2};$$

but a corresponding treatment of curved lines was not possible at the time. The fourteenth century was, in fact, scarcely familiar with conic sections or other curves beyond the line and circle. The problems of the latitude of forms thus failed to inspire the analytic study either of old or of new types of curves. Only in the modern period when the number of recognized curves had been greatly increased, were the notions of variable, function, and rate of change effectively incorporated into mathematics.

During the early modern centuries mathematical activity was devoted in large measure to improvement in arithmetic and algebraic technique and to the recovery of the geometry of the ancients. Here and there one finds some new developments in the theory of curves, but the straight line and the circle continued to play the fundamental role in science as well as in geometry. Copernicus for, example, seems to have felt that the Ptolemaic astronomy was physically impossible because it could not be reconciled with the principle of uniform *circular* motion. However, we are told that the imaginative Nicholas of Cusa had noted the curve traced out by a point on the rim of a cart wheel as the wheel rolled along the road. Although he seems to have been unable to determine its nature or properties, this observation constituted a significant step in the study of curves, for it seems to represent the first modern instance in which a new curve was suggested by natural phenomena. The ancients had invented new curves *ad hoc* to solve specific geometrical problems; they had not discovered these, except for the line and the circle, in the world of nature. The new

curve of Cusanus was followed two centuries later by other curves which were disclosed by, and useful in, the study of physical science.

During the early sixteenth century there were a number of contributions to curve theory. The study of conic sections was revived, especially by the work of Werner, and has ever since occupied a prominent place in mathematics and science. At about the same time, Dürer made significant original additions to the theory of higher curves. He introduced the idea of an asymptotic point and illustrated it by a curve strongly resembling the logarithmic spiral. This curve, later made famous by Jacques Bernoulli, may have been suggested by the revived interest at the time in map construction; it is the plane stereographic projection of the loxodrome on the sphere, and the latter was studied in 1530 by Nuñez. Dürer revived also the ancient kinematic definition of curves, and gave as examples an epicycloid and a new conchoid. Copernicus and Cardan similarly noticed the locus (a straight line) generated by a circle rolling on the inside of another circle with a radius twice as great. However, such work on curves is typical of the time in that it is casual and not systematically developed. Bovelles, for example, noted the cycloid near the beginning of the century, and Galileo referred to it again toward the end, but neither of these men made any headway in determining its equation or properties.

During the first third of the seventeenth century the number of known curves was little greater than it had been two thousand years before. In the decade from 1634 to 1644, however, the situation changed completely. This was the result of the development both of the latent possibilities in methods of curve definition previously adopted and also of new principles which were devised. The cycloid had been noted several times before, but when Mersenne in 1634 and Galileo in 1639 again suggested it as a curve worthy of study, its shape and properties were promptly determined through the composition of motions. This ancient method was supplemented, however, by a powerful new approach—the use of analytic geometry. The new analysis was invented independently by Fermat and Descartes, and somewhat similar methods were familiar at about the same time to Roberval and possibly others. The essence of the new invention, published by Descartes in 1637, lies in the fact that in general an equation in two unknown quantities determines a curve with respect to a set of coordinates in a plane. In a restricted sense, "equations" of curves had been known and used by the ancient Greeks, but the new analysis made them practically the exclusive means of determining curves. As Fermat put it, the equation was the "specific property" of the curve thus defined. This at once makes possible, as Fermat

saw, the definition of an indefinitely great variety of curves; but this tremendous possibility was not at first fully exploited. Descartes, at least, seems rather to have been concerned with the converse of this discovery—that every known plane curve had associated with it, through a coordinate system, an equation in two unknowns. That is, analytic geometry was used at first largely to study *old* curves, the conics especially, rather than to invent *new* ones. Nevertheless, Fermat in 1636 had taken one of the greatest steps in curve theory since the discovery, two thousand years earlier, of the conic sections. He directed attention to an infinite number of plane curves known as the parabolas and hyperbolas of higher degree (or the "parabolas and hyperbolas of Fermat"), given by the equation $y=kx^n$, where n is a positive or negative rational number. This represents the first systematic use of what is without doubt the most important of all methods of curve definition—analytically through equations. Before this time curves had been invented or discovered in connection with certain specialized geometrical problems, and the corresponding analytic forms of expression—the equations of the loci—were attached to them subsequently. Now it was possible to reverse this situation and to inquire into the geometrical analogue or graph of a given equation. The systematic investigation of this latter problem, however, was surprisingly delayed. It was two generations later that Newton, through the publication of his *Enumeratio linearum tertii ordinis*, truly initiated the serious study of higher plane curves defined by equations in two rectangular coordinates.

In the interval between Fermat and Newton curves frequently were defined by other than analytic means. When in the crucial decade from 1634 to 1644 the simple logarithmic and sine curves made their appearance, they were not plotted as the graphs of the equations $y=\sin x$ and $y=\log x$, for the function concept had not been introduced in these situations. The definition was in terms of superimposed motions and of geometrical transformations, ideas developed especially by Roberval and Torricelli, the men to whom it appears we owe the first graphs of these curves.

The method of the composition of movements frequently is ascribed to Galileo, but in physics it goes back at least to the time of Aristotle, and in mathematics it is found still earlier in the quadratrix of Hippias. At about the time that Galileo was applying the method to projectile motion, the study of the cycloid again brought it into prominence in mathematics as a means of defining curves. It may well have been the cycloid which suggested in turn the new logarithm and sine curves. Logarithms had been defined by Napier in terms of velocities of moving points—a line representing the number

decreased with a speed diminishing in geometrical progression while another line representing the logarithms of the number increased with uniform speed. Torricelli seems to have been the first one to modify this idea by imposing these two types of motion upon a single moving point: that is, he took abscissas at equal distances and drew the corresponding ordinates, beginning from a fixed initial ordinate, in continued geometric progression, with ratio less than one. This gave him a monotonically decreasing curve which, because of its form and generation, Torricelli called the *hemhyperbola logaritmica*. He showed that the subtangent of this curve is of constant length, and he found also the area between curve, its asymptote, and a given ordinate.

The composition of motions had been one of the foremost means of defining *new* curves, but Roberval and others extended it to all the higher plane curves then known, as well as to the conic sections. The kinematic tangent methods of Descartes, Roberval, and Torricelli were so widely and effectively applied that they rivaled the analytic methods of Descartes and Fermat. It may well be that the two points of view in the early development of the calculus resulted directly from this rivalry between two methods of curve definition, with the composition of motions leading through Barrow to the fluxions of Newton, and with the analytic view culminating in the differentials of Leibniz.

During the seventeenth century the kinematic and analytic approaches were rivaled, in the discovery of new curves, by a method at once new and old—that of geometric transformation. The Ptolemaic and Mercator projections in the construction of maps, and the methods

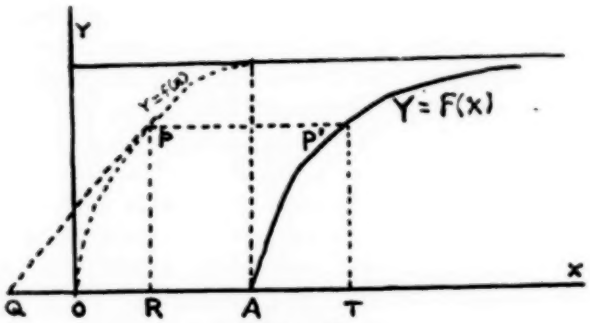

Fig. 2

used by artists (especially Leonardo da Vinci and Dürer) in enlarging or reducing the scale of a design represented simple examples of such transformations. The work of Desargues and Pascal in transforming

conics projectively was but one instance among numerous types of curve transformation widely used at the time. Fermat, for example, in determining areas and tangents for an algebraic curve whose equation contained radicals, first transformed the curve into another whose equation was rational in x and y. Roberval developed a method of transforming a known curve into a new one as follows: Let the curve be $y=f(x)$, let $OA=a$, and let $RQ=s$ be the subtangent for the point P. (See Fig. 2.) Then through the transformation $X=a+s$, $Y=y$ one obtains a new auxiliary curve useful in quadratures.

In determining the quadrature of the cycloid, Roberval had occasion to make use of a new curve defined by a different transformation: Let the circle OCF roll along the line OA and generate the cycloid OPB. (See Fig. 3.) For any point P on the cycloid draw the line DPF parallel to OA, and on this line take $PQ=DF$. The locus of Q is a sine curve which divides the rectangle $OCBA$ into two parts, each equal in area to the generating circle. Since the area $OPBQ$ is equal to the semicircle OCF, the area of $OPBA$ is $3/2$ the area of the generating circle. Roberval did not call the curve OQB a sine curve, but merely the companion of the cycloid. However, it is probable that he was aware of its connection with the trigonometric functions, for in another connection he constructed a curve as follows: Let $Q'Q'_n$ be a line equal in length to a quarter circumference QQ_n of a circle of radius PQ, and let QQ_n and $Q'Q'_n$ be divided into n equal parts. (See Fig. 4.) Let the perpendiculars $P'_iQ'_i$ be taken equal to the sines P_iQ_i of the corresponding points on the quarter circle. The locus of the points P'_i, for n indefinitely large, is obviously a sine curve, although Roberval did not call it by this name. That the first consciously constructed graphs of a trigonometric function were motivated by the geometrical transformation of sine lines rather than by the analytical function concept probably explains the surprising fact that the notion

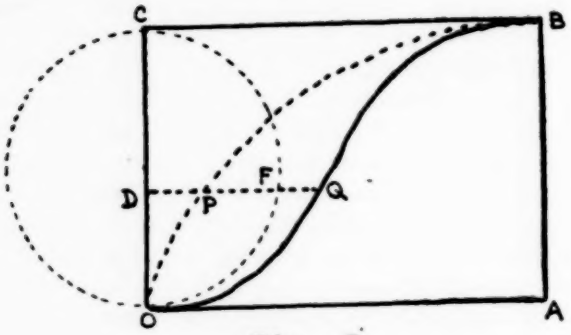

Fig. 3

of periodicity did not enter into the theory of goniometric functions until the following century. De Lagny in 1705 and Cotes in 1722 indicated the periodic nature of the tangent and secant curves; but only after 1748, when Euler's *Introductio in analysin infinitorum* emphasized the multiple angle formulas, was the periodicity of all the trigonometric curves generally recognized.

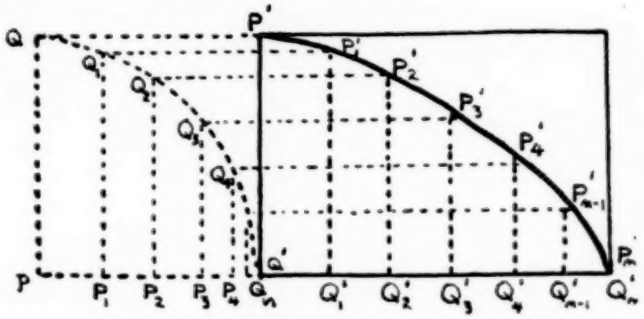

Fig. 4

Several of the new curves discovered in the seventeenth century were found to have striking tangential properties. For the "parabolas of Fermat", $y=kx^n$, the ratio of subtangent to abscissa was found to be $1/n$; and the subtangent of the logarithmic curve was constant. Such discoveries suggested to geometers the possibility of reversing the question—of seeking new curves the tangents of which should possess certain properties specified *a priori*. De Beaune in 1637 sought, in this connection, a curve for which the ratio of the ordinate to the subtangent should for every point be proportional to the difference between the coordinates of the point. This deliberate search for curves exhibiting preassigned tangential properties may be looked upon as a further step in the systematic definition of curves. Moreover, the merging of inverse tangent problems with the kinematic definition of curves did much to inspire Newton's invention of the calculus. The calculus then reciprocally served as a still further means of curve determination, for every differential (or fluxionary) equation in one function of a single independent variable (subject to given boundary conditions) implies a distinctive plane curve.

The investigation of curves with given geometric properties led to the study of those satisfying certain differential equations; and this in turn led to a renewal of the search for curves arising under given physical conditions. Leonardo da Vinci had noted curves of pursuit, and Galileo had speculated on the form of the catenary; but only toward the end of the seventeenth century were such problems

successfully attacked. Then there was a veritable flood of curves and questions arising out of natural phenomena: the tautochrone, the isochrone, the brachistochrone, the tractrix, elastic curves and caustic curves—these and many other curve-problems found their way into mathematics at that time. As an indication of the tendency of the times, Jacques Bernoulli held that geometry should include the study of all curves "which nature herself can produce by simple and expeditious motions". This is a far cry from the restrictive Greek attitude; but the succeeding centuries found need for still further definitions.

The seventeenth century witnessed an extraordinary increase in the number of curves recognized and studied, but there was during this period no general theory of curves beyond those of first and second degree. Descartes had indeed made a beginning in suggesting a distinction between geometrical (or algebraic) and mechanical (or transcendental) curves, according as he was or was not able to write down their equations. But neither Descartes nor Fermat took active part in the popularizing of analytic geometry and the study of curves. The new curves of the time—such as the folium of Descartes, the spiral of Fermat, the pearls of Sluze, the ovals of Cassini, the lemniscate of Bernoulli—were discussed in current learned correspondence; but about the only new curves to appear systematically in treatises of the century were the parabolas and hyperbolas of Fermat.

The general theory of higher plane curves may be said to date from Newton's *Enumeratio* of 1704. This work systematized for the first time practically all plane curves definable by equations of third degree. It was notable also for the fact that curves were plotted for all four quadrants. Moreover, it served as the inspiration during the first half of the eighteenth century of general studies on curve theory which culminated in Euler's *Introductio* of 1748 and Cramer's *Introduction a l'analyse des lignes courbes algebriques*. Such expositions represented belated development of the method of definition in terms of equations which Fermat and Descartes had discovered a century before.

The eighteenth century is to be remembered, in connection with curve definitions, chiefly for its elaboration and perfection of implications inherited from the earlier period. Jacques Bernoulli, for example, had in about 1694 hesitantly proposed the method of polar coordinates; but it was Varignon who, a decade later, actually wrote down the polar equations of a number of curves. Varignon suggested that from known curves new types might be obtained by the simple expedient of interpreting the variables in the rectangular Cartesian representation of the former as polar coordinates for the latter. The

systematic use of polar coordinates followed upon Euler's *Introductio*. That an equation in two variables might represent indefinitely many curves by associating the variables with different types of coordinate systems was thus made clear; but it was the first third of the next century that saw the rise of a variety of coordinate systems, including curvilinear, barycentric, homogeneous, linear, and trilinear coordinates. Finally Cesaro was led by Varignon's idea to the notion of a "natural geometry" based on "intrinsic coordinates".

The method of parametric representation of curves, systematized largely by Euler, was again a development of an implication of a hundred years before. In presenting the fundamental principle of analytic geometry, Descartes had pointed out that for a curve one must have one more unknown than there are conditions or equations. Parametric equations of a curve are only a special case of this fact, but Euler pointed out the great advantage of this form of representation, especially in connection with space curves.

The study of skew curves was largely a product of the eighteenth century, although here again the idea was not new. A few examples of such curves had been given in antiquity and early modern times. Roberval in 1630 had studied a variety of curves drawn on a cylinder, and Jean Bernoulli in 1698 had sought to find maximal and minimal curves drawn on surfaces. Fermat and Descartes had foreseen the possibility of an analytic geometry of three dimensions, and La Hire again mentioned this in 1679. The first application of it to curves and surfaces was made by Parent in 1700, but this memoir went largely unnoticed. The first influential treatment of the geometry of space curves was made in 1731 by the young Clairaut in his celebrated *Recherches sur les courbes à double courbure*, and the classic treatment was given about the middle of the century by Euler.

Possibly the most important general contribution of the eighteenth century to mathematics was the development of the notion of function. This idea again had been familiar to earlier men, from Descartes and Fermat to Newton and Leibniz; but it was Euler who established it firmly in his *Introductio*. Improving on the definition given fifty years earlier by Jean Bernoulli, Euler wrote: "A function of a variable quantity is an analytic expression composed in any manner whatever of this variable and constants". In this definition Bernoulli and Euler had in mind the chief means of defining plane curves—by means of "analytic expressions" or equations in two variables; but the emphasis, as in the case of Descartes, was on the equation rather than on the curve. On at least one occasion, nevertheless, Euler inverted the situation and used the term function to designate the relationship between x and y implied by any curve "drawn free-hand" in the xy plane.

The broader view of Euler, which would make any traceable curve whatsoever a proper object of mathematical study, met with little favor in the prosy eighteenth century, but it was the basis for much of the original development in curve definition throughout the hundred years following. Fourier, for example, was led by it in 1807 to his trigonometric representation of graphically-given and ill-behaved curves. A little later Dirichlet generalized it in his own well-known definition: y is said to be a function of x whenever a precise law of correspondence between x and y is given. This represented a new step in the history of curves for it introduced the idea of the determination of curves through functional correspondence, thus extending the Cartesian definition by means of equations. For example, the "Dirichlet function"—$y=f(x)$, where y is 1 or 0 according as x is rational or irrational—presented mathematicians with a "curve" which is totally discontinuous. They began to feel uneasy about the nature and properties of curves in general. That a continuous curve must have a tangent at all but a few exceptional points had been widely accepted, and Ampère believed that he had proved this in 1806. That this presumption, suggested by crude experience, is totally incorrect was shown by Bolzano in 1834 when he drew the first curve which was continuous throughout an interval and yet had no tangent at any point in this interval. Even in antiquity mathematicians had used infinite processes to approximate to a given familiar curve; but Bolzano inverted this situation and used infinite processes to define his new tangentless curve. He did this as follows: Let PQ be a line segment inclined to the horizontal. (See Fig. 5.) Let it be bisected

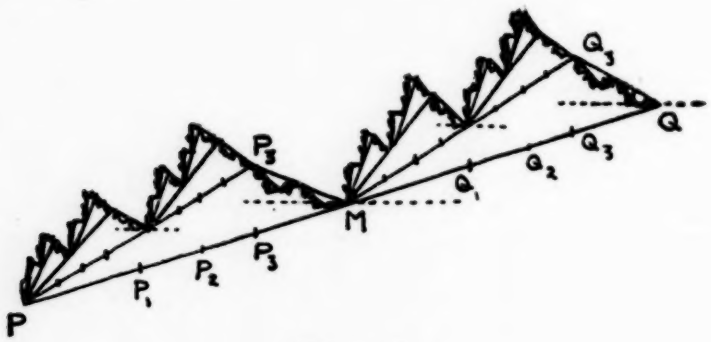

Fig. 5

by the point M, and subdivide the segments PM and MQ into four equal parts, the points of division being P_1,P_2,P_3 and Q_1,Q_2,Q_3. Let P'_3 be the reflection of P_3 in the horizontal through M, and Q'_3 be the

reflection of Q_3 in the horizontal through Q, and form the broken line $PP'_3MQ'_3Q$. Now repeat for each of the four segments of this broken line the operation described above for the segment PQ, obtaining in this manner a broken line of 4^2 segments. Continuing this process indefinitely, the broken-line figure will converge toward a new curve which is continuous but without a tangent at any point.

The curve of Bolzano was generally overlooked, but later in the century the illustration of a continuous non-differentiable function given by Weierstrass initiated a general study of curves without tangents and led to the recognition of a new class, the so-called *ausserordentlichen* or crinkly curves, with countless oscillations in every finite interval. Gradually it came to be recognized that of all continuous curves, those which are so well-behaved as to have tangents are the exceptions rather than the rule. Still other odd properties of curves were also found—space-filling curves, and closed curves without interiors or exteriors. In the minds of such men as Weierstrass and Cantor the question arose, "What, after all, is a curve?" Up to the middle of the nineteenth century the concept of curve in general had been more or less taken for granted; it had been only specific curves that had called for definition to distinguish them from others.

The search for adequate descriptions of a curve took two general directions: one began with a simple geometric characteristic of most curves, their continuity, and this led to the definition of Cantor; the other started from the analytic viewpoint of curves as defined by equations, and this led to the definition of Weierstrass. The definitions of the word curve given by these men were in a sense by-products of their attempts to furnish analysis with a sound arithmetic basis. Cantor proceeded from a study of the continuum to *Mengenlehre* or point-set theory. A "curve in the sense of Cantor" is essentially a closed well-ordered ensemble of points perfect within itself: being given any two points of the set, one can establish between these two points a chain of points of the set at distances successively less than any given number. This is analogous to the notion of the straight line as a measure of distance, and so is practically equivalent geometrically to the definition of a continuous curve as a closed esnemble of points which is the image, through a one-to-one correspondence, of a segment of a straight line.

In the more restricted definition of curve given by Weierstrass, the geometric character is lost. An analytic curve is simply the totality of points whose coordinates are analytic functions of a single parameter. It represents, except for the inclusion of complex coordinates, essentially a refinement of the idea of definition of Fermat and Descartes. The requirement, in the definition of Weierstrass, that the

functions be differentiable as often as may be required was felt by mathematicians to be too strong. Jordan therefore modified it, toward the close of the century, to correspond more closely to the kinematic definition of a curve as the trajectory of a point. Jordan required only that, for a continuous plane curve, the defining functions $x=f(t)$, $y=g(t)$ be continuous in the real variable t as t describes a closed interval $a\leq t\leq b$. But he found that this flexible definition included space-filling curves, and to exclude them he required that the curve have no double-point within the interval ab—that is, for $a<t<b$, if $f(t_1)=f(t_2)$ and $g(t_1)=g(t_2)$, then $t_1=t_2$. He then demonstrated the important separation theorem that if such a "simple Jordan curve" is closed—that is, if $f(a)=f(b)$ and $g(a)=g(b)$—then it possesses one of the most pronounced empirical notions of a closed plane curve, namely, it divides the plane into two regions, an interior and an exterior. However, unless one places further restrictions on the functions f and g, a Jordan curve does not necessarily have other properties suggested by experience; it may not, for example, have tangents, length, or curvature. And so the search for curve definitions continues.

A survey of the historical stages in the definition of curves impresses one with a sense of the progressive development in mathematics. Nor is it likely that this progress in curve theory will soon reach an end. The twentieth century already has made a notable contribution to the definition of curves; it has come to the Socratic realization, so well expressed by Klein, that "everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions".

Brief Notes and Comments

Edited by
MARION E. STARK

13. The method of handling trigonometric problems described on pages 150-1 of the December issue depends so much on reading formulas from the diagram, that it seems desirable to exhibit better drawings for Figures 2 and 3.

Fig. 2

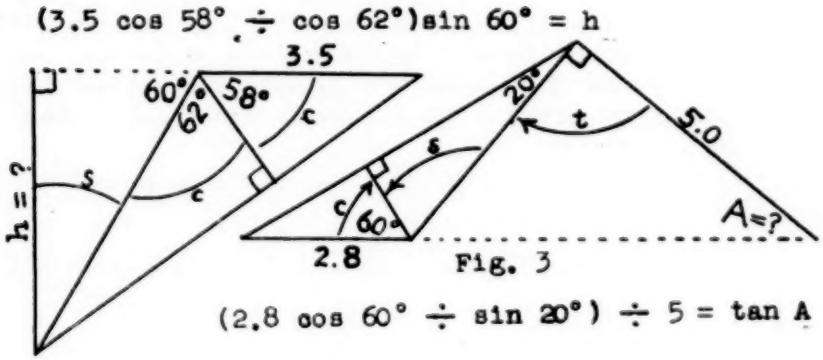

Fig. 3

In Fig. 2, start with 3.5, following the arrow indicates multiplication by the cosine, backing along the next arrow indicates division by the cosine, and following the next arrow indicates multiplication by the sine, and brings the computer to the value of h without requiring pauses at way-stations to write down ratio defintions of the functions.

Similarly in Fig. 3, starting with 2.8, following one arrow and backing along the next indicates multiplying by a cosine, then dividing by a sine, bringing the computer to the side at the head of the t arrow: this divided by the side at its other end is indicated as the tangent of the desired angle.

Tufts College. WILLIAM R. RANSOM.

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14. *Successive Derivatives of tan x.* This note offers a convenient way to get the successive derivatives of tan x, as needed in its expansion by Maclaurin's series.

We have $f(x) = \tan x$

$$f'(x)=\sec^2x=1+\tan^2x$$
$$f''(x)=2\tan x\sec^2x=2\tan x+2^2\tan^3x$$
$$f'''(x)=2\sec^2x+2\cdot 3\tan^2x\sec^2x$$
$$=2+(2+2\cdot 3)\tan^2x+2\cdot 3\tan^4x, \text{ etc.}$$

A term $a_k\tan^kx$ in $f^{(n)}(x)$ gives $f^{(n+1)}(x)$ $k\ a_k\tan^{k-1}x+ka_k\tan^{k+1}x$. Except at the ends there is another term with each of these exponents, so the coefficient of $\tan^{k-1}x$ in $f^{(n+1)}(x)$ is $(k-2)\ a_{k-2}+ka_k$, with $a_{k-2}=0$ when $k\leqq 1$, and $a_k=0$ when $k\geqq n+2$.

Then the process can be arranged as in the table below. The numbers across the top are the exponents of tan x. The numbers in the left hand column are orders of derivatives. A number in the body of the table not in parentheses is the coefficient of the derivative in its row and the power of tan x in its column. A number in parentheses is the product of the number above it by the exponent at the top. Any coefficient is the sum of the numbers in parentheses in the row above immediately to left and right. The coefficients in the 0 column are the wanted values of $f^{(n)}(0)$.

If the computation is carried out by diagonals as shown no unused numbers are computed.

	0	1	2	3	4	5	6
0		1					
		(1)					
1	1		1				
	(0)		(2)				
2		2		2			
		(2)		(6)			
3	2		8		6		
	(0)		(16)		(24)		
4		16		40		24	
		(16)		(120)		(120)	
5	16		136		240		120
	(0)		(272)		(960)		(720)
6		272		1232		1680	
		(272)		(3696)		(8400)	
7	272		3968		12096		
	(0)		(7936)		(48384)		
8		7936		56320			
		(7936)		(168960)			
9	7936		176896				
	(0)		(353792)				
10		353792					
		(353792)					
11	353792						
	(0)				Etc.		

Worcester Polytechnic Institute. RAYMOND K. MORLEY.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 551. Proposed by *W. V. Parker*, Louisiana State University.

Two given straight lines, s and s', intersect in O. Take A, B, on s on the same side of O, and A', B' on s', on the same side of O, such that $OA = OA'$, $OB = OB'$. Draw $AP \perp BP$, and $A'P' \| AP$ and $B'P' \| BP$. Show that (area $OPP')^2 =$ area $OAA' \cdot$ area OBB'.

I. Solution by *J. S. Guérin*, Catholic University of America.

Stewart's Theorem applied to P and the collinear points O, A, B gives

$$\overline{AP}^2 \cdot OB = \overline{OP}^2 \cdot AB + \overline{BP}^2 \cdot OA - OA \cdot OB \cdot AB,$$

or, since $OB = OA + AB$ and $\overline{AB}^2 = \overline{AP}^2 + \overline{BP}^2$,

$$\overline{AP}^2(OA + OB) = AB(\overline{OP}^2 - \overline{OA}^2). \qquad \text{Similarly,} \tag{1}$$

$$\overline{A'P'}^2(OA + OB) = AB(\overline{OP'}^2 - \overline{OA}^2). \tag{2}$$

Hence, upon dividing (1) by (2),

$$\overline{AP}^2/\overline{A'P'}^2 = (\overline{OP}^2 - \overline{OA}^2)/(\overline{OP'}^2 - \overline{OA}^2). \tag{3}$$

Let M, M' be the midpoints of AP and $A'P'$, and H, H' be the orthogonal projections of O on AP, $A'P'$, respectively. (3) then gives

$$\overline{AP}^2/\overline{A'P'}^2 = 2AP \cdot MH/2A'P' \cdot M'H',$$

and hence

$$AP : A'P' = MH : M'H'.$$

Thus the lines PP', AA' and OHH' are concurrent in, say W. Then

$$\text{area } OPP' = \text{area } OWP' - \text{area } OWP$$
$$= OW(P'H' - PH)/2 = (OW \cdot CP')/2,$$

C being the intersection of BP and $A'P'$. Now let a perpendicular from O intersect AA' in D and BB' in E. The angles OWD and PBB' included between parallel lines are equal and one has

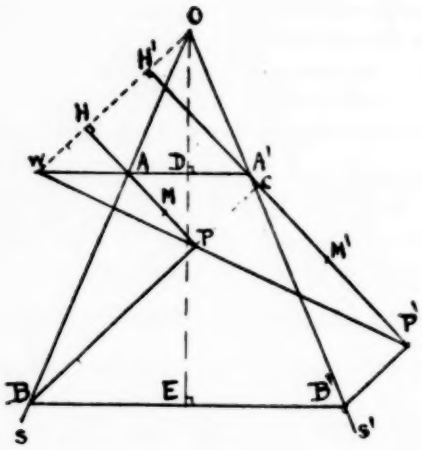
$$CP' = BB' \sin PBB', \quad OW = OD/\sin PBB'.$$

Hence $\quad$ area $OPP' = OW \cdot CP'/2 = OD \cdot BB'/2.$

Furthermore $\quad$ Area $OAA' = OD \cdot AA'/2$, $\quad$ area $OBB' = OD \cdot BB'/2$

and $\quad OD : OE = AA' : BB'.$

The equality to be proved is readily derived from the last three relations.



II. Solution by *Howard Eves*, Syracuse University.

The problem is unnecessarily restricted. We shall establish the result without the condition $AP \perp BP$. Let s and s' be axes in an oblique cartesian coordinate system. Give A,B,A',B' the coordinates $(a,0)$, $(b,0)$, $(0,a)$, $(0,b)$. We may then take $AP : x+my-a=0$, $A'P' : x+my-ma=0$, $BP : x+ny-b=0$, $B'P' : x+ny-nb=0$. We readily find

$$P : \left(\frac{an-bm}{n-m}, \frac{b-a}{n-m} \right), \qquad P' : \left(\frac{mn(a-b)}{n-m}, \frac{bn-am}{n-m} \right).$$

Hence

$$\begin{vmatrix} \dfrac{an-bm}{n-m} & \dfrac{b-a}{n-m} \\ \dfrac{mn(a-b)}{n-m} & \dfrac{bn-am}{n-m} \end{vmatrix}^2 = a^2b^2 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \cdot \begin{vmatrix} b & 0 \\ 0 & b \end{vmatrix},$$

which proves the theorem, since the area of triangle (x_1,y_1), (x_2,y_2), (0,0) is

$$k\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix},$$

where k is a constant depending only on the angle between the coordinate axes.

Also solved by *H. E. Fettis*.

No. 555. Proposed by *Walter B. Clarke*, San Jose, California.

A transversal meets the sides BC, CA, AB of a triangle ABC in the points P, Q, R; the lines AP, BQ, CR meet the circumcircle (O) of ABC in the points P', Q', R'; the lines $A'P'$, $B'Q'$, $C'R'$ joining P', Q', R' to the diametric opposites of A, B, C on (O) meet BC, CA, AB in the points A_0, B_0, C_0. Show that the lines AA_0, BB_0, CC_0 are concurrent.

I. Solution by *J. S. Guérin*, Catholic University of America.

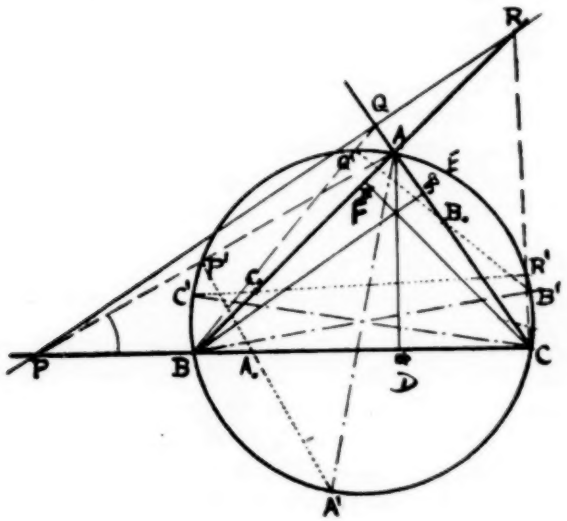
Let D, E, F, on BC, CA, AB respectively, be the feet of the altitudes of the triangle ABC. The relation

$$PB\cdot PC = PP'\cdot PA = PA_0\cdot PD \qquad \text{gives}$$

$$PC/PD = PA_0/PB \tag{1}$$

or
$$(PC-PA_0)/(PD-PB) = PA_0/PB$$

or
$$(PA_0-PB)/(PD-PB) = PB/PD,$$



which reduce to

$$A_0C/DB = PA_0/PB \quad \text{and} \quad A_0B/DC = PB/PD.$$

These relations may be combined as

$$A_0B/A_0C = (PB \cdot PB \cdot DC)/(PD \cdot PA_0 \cdot DB),$$

which, by another application of (1), reduces to

$$A_0B/A_0C = (PB \cdot DC)/(PC \cdot DB).$$

Similarly,
$$B_0C/B_0A = (QC \cdot EA)/(QA \cdot EC),$$
$$C_0A/C_0B = (RA \cdot FB)/(RB \cdot FA).$$

Multiplication of these last three equations results in

$$\frac{A_0B}{A_0C} \cdot \frac{B_0C}{B_0A} \cdot \frac{C_0A}{C_0B}$$

$$= \frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB} \cdot \frac{DC}{DB} \cdot \frac{EA}{EC} \cdot \frac{FB}{FA} = -1,$$

which proves the proposition.

II. Solution by *Nev. R. Mind.*

The cevian AA_0 subtends a right angle at the point P', hence the circle (AA_0) having AA_0 for diameter passes through P'. Thus the point P is the trace, on the side BC, of the radical axis AP' of the circle (AA_0) with the circumcircle (O) of the triangle ABC.

Similarly for the points Q and R. Now the points P, Q, R are collinear, by assumption, hence the three cevians AA_0, BB_0, CC_0 are concurrent (N. A. Court, *On the cevians of a triangle*, this MAGAZINE, Vol. XVIII, No. 1, October, 1943, p. 3, Art. 4).

Moreover, the point common to the three cevians AA_0, BB_0, CC_0 is the isotomic point, with respect to the triangle ABC, of the pole of the line PQR with respect to the polar circle (H) of ABC. (*ibid.*, Art. 3b).

III. Solution by *Howard Eves*, Syracuse University.

Let us prove the following projective generalization: *Given a triangle ABC inscribed in a conic Σ. Let O be any point and let AO, BO, CO cut Σ again in A', B', C'. Let a transversal meet BC, CA, AB in P, Q, R; and let AP, BQ, CR cut Σ again in P', Q', R'. Finally, let $A'P'$, $B'Q'$, $C'R'$ intersect BC, CA, AB respectively in the points A_0, B_0, C_0. Then AA_0, BB_0, CC_0 are concurrent.*

We shall use projective coordinates, taking triangle ABC as triangle of reference and O as unit point; $A : (1,0,0)$, $B : (0,1,0)$, $C : (0,0,1)$, $O : (1,1,1)$. Then the equation of Σ has the form

$$Ux_2x_3+Vx_3x_1+Wx_1x_2=0,$$

and we take, where u, v, w are appropriate constants,

$$uUx_1+vVx_2+wWx_3=0$$

as the equation of the transversal PQR. We now readily find, in order,

$$P : (0, -vV, wW),\ P' : [-vwU, -vV(v-w), wW(v-w)],$$

$$A' : (-U, V+W, V+W),\ A_0 : (0,v,w).$$

We similarly find $B_0 : (u,0,w)$, $C_0 : (u,v,0)$.

Therefore AA_0, BB_0, CC_0 are concurrent at the point (u,v,w), and the theorem is established.

No. 556. Proposed by *Howard D. Grossman*, New York City.

Prove that the limit, as $n\to\infty$, of the ratio

$${}_{(n+1)(p+q)}C_{(n+1)p} : {}_{n(p+q)}C_{np}$$

is $(p+q)^{p+q}/p^pq^q$.

Solution by *H. J. Zimmerberg*, University of Chicago.

By definition this ratio is equal to

$$\frac{[(n+1)(p+q)]!}{[(n+1)p]![(n+1)q]!}\cdot\frac{(np)!\,(nq)!}{[n(p+q)]!},$$

which can be written as

$$\frac{[(n+1)(p+q)]\cdots[n(p+q)+1]}{[(n+1)p]\cdots[np+1][(n+1)q\cdots(nq+1)]}.$$

As the numerator and denominator contain $p+q$ terms each we can divide each term by n. Passing to the limit we obtain the required result.

Also solved by *A. B. Farnell* and *Howard Eves*.

No. 558. Proposed by *Frank C. Gentry*, University of New Mexico.

A variable triangle has a fixed base and the difference of its base angles is constant. Show, both analytically and synthetically, that

the locus of the third vertex and the locus of the orthocenter are identical.

Solution by *Howard Eves*, Syracuse University.

(a) Let ABC be the triangle, A and B fixed, C variable. Let H be the orthocenter. Then

$$\measuredangle HAB - \measuredangle HBA = (90° - \measuredangle B) - (90° - \measuredangle A) = \measuredangle A - \measuredangle B.$$

Hence the loci of C and H coincide.

Since the pencils $A(C)$ and $B(C)$ are homographic we know that the locus is a conic. A little investigation shows that the locus possesses two infinite points in perpendicular directions; hence the conic is a rectangular hyperbola.

(b) The cartesian equation of the locus of C may readily be found as follows. Take AB as x-axis and the perpendicular bisector of AB as y-axis. Let the coordinates of A, B, C be $(-a,0)$, $(a,0)$, (x,y). respctively. Then

$$\tan A = y/(a+x), \quad \tan B = y/(a-x),$$

whence $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} = \frac{-2xy}{a^2-x^2+y^2} = k.$$

The equation of the locus is, therefore,

$$k(x^2-y^2-a^2) - 2xy = 0, \tag{1}$$

where $k = \tan(A-B)$, a constant. This is a rectangular hyperbola having its center at the origin and such that its axes make angles of $\frac{1}{2}(A-B) \pm 45°$ with the x-axis. Also, the hyperbola passes through A and B.

Temporarily taking the coordinates of C as (m,n) we readily find the coordinates of H to be

$$\left(m, \frac{a^2-m^2}{n} \right).$$

Substituting these in the left member of (1) we get, upon reduction,

$$[(a^2-m^2)/n^2][k(m^2-n^2-a^2-2mn].$$

This vanishes since C is on the curve (1). Therefore H is also on the curve (1).

Also solved by *J. S. Guérin*.

No. 571. Proposed by *Pvt. Nicholas Kushta*, University of Illinois.

Determine the the limits of the following functions as $x \to \infty$.

(a) $e^x - x$, (b) $x^2 - \csc^2(1/x)$.

Solution by *W. S. Loud*, Massachusetts Institute of Technology.

(a) As $x \to +\infty$, $e^x/x \to +\infty$, by l'Hospital's rule; as $x \to -\infty$, $e^x/x \to 0$. Therefore

$$\lim_{x \to +\infty} e^x - x = \lim_{x \to +\infty} x \cdot \lim_{x \to +\infty} (e^x/x - 1) = +\infty \cdot +\infty = +\infty,$$

$$\lim_{x \to -\infty} e^x - x = \lim_{x \to -\infty} x \cdot \lim_{x \to -\infty} (e^x/x - 1) = -\infty \cdot -1 = +\infty.$$

(b) There is no need to distinguish between $+\infty$ and $-\infty$. With $u = 1/x$, the desired limit becomes

$$\lim_{x \to \infty} \frac{\sin^2 u - u^2}{u^2 \sin^2 u}.$$

Repeated applications of l'Hospital's rule give

$$\lim_{u \to 0} \frac{\sin^2 u - u^2}{u^2 \sin^2 u} = \lim \frac{\sin 2u - 2u}{2u \sin^2 u + u^2 \sin 2u}$$

$$= \lim_{u \to 0} \frac{2 \cos 2u - 2}{2 \sin^2 u + 4u \sin 2u + 2u^2 \cos 2u}$$

$$= \lim_{u \to 0} \frac{-4 \sin 2u}{6 \sin 2u + 12u \cos 2u - 4u^2 \sin 2u}$$

$$= \lim_{u \to 0} \frac{-8 \cos 2u}{24 \cos 2u - 32u \sin 2u - 8u^2 \cos 2u}.$$

This last form is not indeterminate and yields the result, $-\frac{1}{3}$.

The result could have been obtained by use of Taylor's theorem. With $\sin^2 u = u^2 - u^4/3 + \cdots$, we have

$$\lim_{u \to 0} \frac{\sin^2 u - u^2}{u^2 \sin^2 u} = \lim_{u \to 0} \frac{-u^4/3 + 0(u^6)}{u^4 + 0(u^6)} = \lim_{u \to 0} \frac{-1/3 + 0(u^2)}{1 + 0(u^2)} = -\tfrac{1}{3},$$

where $O(x)$ is an infinitesimal of the same order as x.

Also solved by *Howard Eves, J. S. Guérin, L. M. Kelly, A. Sisk, P. D. Thomas, H. J. Zimmerberg*, and the *Proposer*.

No. 572. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A uniform chain of length L and total mass M is held vertically with its lower end just touching a platform balance. The fixed upper end is released and the chain accumulates on the scale pan. What is the maximum reading of the balance?

Solution by *Frank Hawthorne*, New Rochelle, New York.

The force on the balance consists of two parts: F_1, the impulse imparted to the pan because of the change of momentum of the links striking the pan, and F_2, the weight of the chain already accumulated on the pan. Now we have

$$F_1 dt = v dm = Mv dL/L, \qquad F_1 = (Mv/L)(dL/dt) = Mv^2/L.$$

Since each particle strikes the pan with a velocity due only to free fall, $v^2 = 2gS$, where S is the part of L which has accumulated on the pan. Thus, $F_1 = 2MgS/L$. Since F_2 is evidently MgS/L, the total force is

$$F_1 + F_2 = 3MgS/L,$$

which has the maximum value $3Mg$ at the instant when $S = L$. Thus, the maximum apparent weight is three times the actual weight and occurs just as the last of the chain strikes the pan.

Also solved by the *Proposer*.

No. 573. Proposed by *Fred G. Fender*, Bloomfield, N. J.

A summer camper rowed upstream one mile when his hat blew off into the water beside him. As it was an old hat he decided to let it go. Ten minutes later he remembered that he had put his return ticket under the hatband. Rowing at the same rate as before, he just reached the hat (and the ticket) at the starting point. How fast was the stream flowing?

Solution by *W. T. Short*, Oklahoma Baptist University.

On the water surface, the rate of departure from the hat and the rate of approach to the hat will be the same. Thus the camper was away from the hat ten minutes in departing and (assuming no loss of time in turning) ten minutes in returning. In twenty minutes the hat drifted one mile downstream. Hence the rate of the stream was one mile in twenty minutes or three miles per hour.

Also solved by *R. G. Blake, M. I. Chernofsky, W. E. Cox, N. E. Dodson, Howard Eves, H. S. Grant, H. D. Grossman, Frank Hawthorne,*

C. S. Larkey, W. S. Loud, Nev. R. Mind, Howard Perry, P. D. Thomas, H. J. Zimmerberg, the *Proposer,* and one whose sheet was unsigned. Many of the solvers noted that the result is independent of the rate of rowing.

No. 576. Proposed by *P. T. Thomas,* U. S. Navy.

$$I = \int \frac{a \cosh x + b \sinh x}{c \cosh x + d \sinh x} dx.$$

Solution by *H. J. Zimmerberg,* University of Chicago.

We first determine numbers A and B such that

$$a \cosh x + b \sinh x = A(c \cosh x + d \sinh x) + B(c \sinh x + d \cosh x).$$

The solutions, provided $c^2 \neq d^2$, are

$$A = (ac - bd)/(c^2 - d^2), \quad B = (bc - ad)/(c^2 - d^2).$$

Hence, if $c^2 \neq d^2$, the integral I has the value

$$I = Ax + B \ln(c \cosh x + d \sinh x) + K,$$

where K is the constant of integration.

If $c^2 = d^2$, the integration may be effected by multiplying numerator and denominator of the integrand by $c \cosh x - d \sinh x$. Since the denominator then reduces to c^2, we have

$$I = \int \{a/c + [(ac - bd)/c^2] \sinh^2 x + [(bc - ad)/2c^2] \sinh 2x\} dx$$
$$= [(ac + bd)/2c^2] x + [(ac - bd)/4c^2] \sinh 2x + [(bc - ad)/4c^2] \cosh 2x + K.$$

If the hyperbolic functions are replaced by exponentials, this result may be written

$$I = (a \pm b)x/2c - (a \mp b)e^{\mp 2x}/4c + K, \quad \text{where} \quad d = \pm c.$$

Also solved by *C. B. Boyer, A. B. Farnell, H. E. Fettis* and the *Proposer.*

No. 577. Proposed by *Emory P. Starke,* Rutgers University.

Across one corner of a rectangular room two 4-foot screens are placed in such a manner as to enclose the maximum floor space. Determine their positions.

I. Solution by *A. Sisk,* Maryville, Tennessee.

With the walls as axes and the corner as origin, let P, the vertex of the quadrilateral opposite O, have coordinates (x,y). Then, as seen from a figure, the area of the inclosed floor space is given by

$$A = xy + \tfrac{1}{2}y\sqrt{16-y^2} + \tfrac{1}{2}x\sqrt{16-x^2}.$$

Then $A'_x = y - (x^2-8)/\sqrt{16-x^2}$, $A'_y = x - (y^2-8)/\sqrt{16-y^2}$.

Therefore $A'_x = A'_y = 0$ if

$$y = (x^2-8)/\sqrt{16-x^2}, \qquad x = (y^2-8)/\sqrt{16-y^2}.$$

To eliminate x between these equations and solve for y will give the same equation as to eliminate y and solve for x. Therefore the values of x and of y which make $A'_x = A'_y = 0$ are equal. Hence solve the equation

$$x\sqrt{16-x^2} = x^2 - 8 \quad \text{or} \quad x^4 - 16x^2 + 32 = 0,$$

and get $$x^2 = 8 + 4\sqrt{2}, \qquad x = \sqrt{8+4\sqrt{2}} = 3.7 \text{ ft.}$$

(Since $x,y > 0$, the solution $x^2 = 8 - 4\sqrt{2}$ must be discarded.) It is then easy to compute the distance from the corner to where each screen touches the wall. This is $\sqrt{16+8\sqrt{2}} = 5.2$ ft., as is also the distance OP. Hence each screen is the base of an isosceles triangle with vertical angle 45° at O. The maximum area enclosed is $8+8\sqrt{2} = 19.3$ sq. ft.

II. Solution by the *Proposer*.

This problem has a special interest because its solution is possible by means of elementary geometry alone.

It is easily seen that a triangle with fixed base and given opposite angle has maximum area when the other sides are equal. (The locus of the vertex of the given angle is a circular arc having the fixed base as its chord: the most distant point on this arc from the base is its midpoint.) Let B, C be the points at which the screens touch the respective walls. For a fixed length of BC, the triangle BPC has a constant area, and the triangle OBC is maximum, by our proposition, when $OB = OC$. Thus, whatever the value of BC, the maximum quadrilateral $OBPC$ occurs when $OB = OC$. Since OBP and OCP are congruent, each must be then a maximum. But $BP = 4$ ft. and angle A equals 45°, whence, again using the proposition stated at the outset, we have $OB = OP$ and $OP = OC$. Finally angle B is 67.5° and the maximum area of $OBPC$ comes out $8(\sqrt{2}+1)$ sq. ft.

III. Solution by *F. Hawthorne*, New Rochelle, New York.

The proposed figure is a quadrangle having one right angle and such that the two sides which meet in the vertex opposite the right angle are equal. Now it is possible to fit together four congruent quadrangles of this form so as to form an equilateral octagon. As is

well know, if the octagon is to have a maximum area for a given length of side, it must be regular. Hence, for maximum area, one such quadrangle will be one-fourth of a regular octagon.

It follows immediately that the screens should be so placed that each forms, with one wall and the bisector of the right angle at the corner of the room, an isosceles triangle of which the screen is the base.

Also solved by *A. B. Farnell* and *Edith L. Morgan*.

EDITORIAL NOTE. In such a problem as this an appeal to symmetry is often made in an attempt to simplify the work and—as here—reduce the quantity to be maximized to a function of a single variable. In all three solutions given above, the symmetry is established by actual proof. It should be noted, however, that symmetry alone is not a valid argument. See, for example, problem No. 585 (this MAGAZINE, January, 1945) for which the point which minimizes the required distance is not on a line of symmetry. The appeal to symmetry is valid however, if it can be established that a unique maximum (or minimum) exists.

PROPOSALS

No. 597. Proposed by *H. J. Zimmerberg*, University of Chicago.

Integrate $$\int \frac{a \sin^3 x + b \cos^3 x}{c \sin x + d \cos x} dx.$$

No. 598. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once and only once, form three numbers, m, n, r such that the sixth power of r equals the sum of m and n.

No. 599. Proposed by *H. S. Grant*, Rutgers University.

Prove the following summations; n being an odd integer:

(a) $$1 - \frac{n^2-1^2}{2!} + \frac{(n^2-1^2)(n^2-3^2)}{4!} - \cdots = \pm n,$$

(b) $$1 - \frac{n^2}{2!} + \frac{n^2(n^2-2^2)}{4!} - \cdots = 0.$$

No. 600. Proposed by *N. A. Court*, University of Oklahoma.

Three concurrent non-coplanar lines, bearing the equal segments SA, SB, SC, are cut in the points A', B', C' by a variable plane $PA'B'C'$ passing through a fixed point P. Find the locus of the points common to the three variable spheres having A', B', C' for centers and AA', BB', CC' for radii.

No. 601. Proposed by *E. P. Starke*, Rutgers University.

There is reason to suppose that in the long run and with a very large number of pupils in a properly instructed and properly tested mathematics class there would be approximately $12\frac{1}{2}\%$ who failed to attain a satisfactory grade. Because she wants to do everything in the most modern, scientific manner, Miss X adjusts her grades in each class of 24 pupils so that there are always exactly 3 failures (and the approved number of A's, B's, etc.)

Assuming that $12\frac{1}{2}\%$ is the correct probability of a student's passing and that Miss X teaches and tests satisfactorily, show that, on the average, less than one class (24 pupils) in four should have exactly three failures each. How often should Miss X expect a class in which half or more of the pupils ought to fail?

No. 602. Proposed by *V. Thébault*, Tennie, Sarthe, France.

The tangents MA, MB from a point M to a parabola (P) meet the curve in the points A, B. Find the locus of the point M, if the circle MAB passes through the vertex of (P).

No. 603. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Construct a trapezoid $ABCD$ given its diagonals and its non-parallel sides.